MATH30002/MATH0009: Topics in Discrete Maths.
Error-Correcting codes: Practice exam question

1. (a) Let C be an $[n, k, d]$-linear code over $\mathbb{F}_p$.

   i. **(2 marks)** Explain what it means for a matrix G to be a generator matrix for C. How many rows/columns does G have?

   ii. **(2 marks)** Prove that $|C| = p^k$.

   iii. **(3 marks)** Let C be the linear code over $\mathbb{F}_2$ with generator matrix:

$$G = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}.$$

   Write down the parameters n, k, d.

   iv. **(3 marks)** Find a parity check matrix for C.

   (b) Let C be an (n, M, d)-code over a finite alphabet F.

   i. **(2 marks)** Define the Hamming distance on F^n and the Hamming sphere $S(\mathbf{v}, r)$ for $\mathbf{v} \in F^n$ and $r \geq 0$.

   ii. **(4 marks)** Show that $|S(\mathbf{v}, r)| = \sum_{m=0}^{r} \binom{n}{m} (|F| - 1)^m$ for $\mathbf{v} \in F^n$ and $r \geq 0$.

   iii. **(4 marks)** Show that $S(\mathbf{c}_i, t) \cap S(\mathbf{c}_j, t) = \phi$, where $t = \lfloor \frac{d-1}{2} \rfloor$.

   iv. **(5 marks)** State and prove the sphere packing bound for C.

   (c) i. **(3 marks)** Let C be an $[n, k, 5]$-linear code over $\mathbb{F}_3$. If C is perfect then show that $1 + 2n^2 = 3^{n-k}$.

   ii. **(7 marks)** Show that there cannot exist a perfect $[2m, m, 5]$-linear code over $\mathbb{F}_3$ for any $m \geq 1$.

   (d) Let C_1, C_2 be linear codes over $\mathbb{F}_p$ with parameters $[n_1, k_1, d_1]$ and $[n_2, k_2, d_2]$ respectively. The (external) direct sum code is $C_1 \oplus C_2$, i.e. the $\mathbb{F}_p$-vector space of pairs $(\mathbf{c}_1, \mathbf{c}_2)$ with $\mathbf{c}_1 \in C_1$ and $\mathbf{c}_2 \in C_2$.

   i. **(3 marks)** Write down the parameters n, k, d for this code in terms of the ones for C_1 and C_2.

   ii. **(5 marks)** Show for $p = 2$:

$$W_{C_1 \oplus C_2}(x, y) = W_{C_1}(x, y) W_{C_2}(x, y).$$

   iii. **(2 marks)** Hence show that:

$$W_{(C_1 \oplus C_2)^\perp}(x, y) = W_{C_1^\perp \oplus C_2^\perp}(x, y).$$

   (e) i. **(2 marks)** Write down the standard parity check matrix for the Hamming code $\mathbf{Ham}_4$. How many errors can this code detect/correct? Justify your answer.

   ii. **(3 marks)** Suppose you receive the message $\mathbf{v} = 000110001000011$. Show that at least one error has been made and correct it (assuming exactly one error has been made).