The Leech lattice and two remarkable results.

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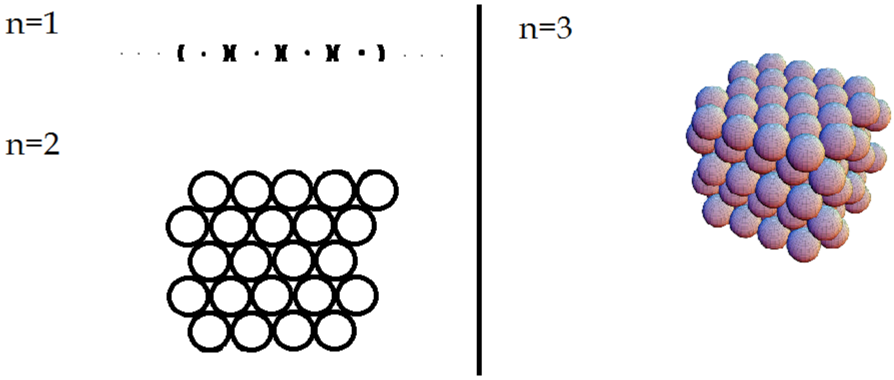
A natural question to ask in geometry is the following:

"What is the most efficient way to pack n-dimensional spheres in $\mathbb{R}^n$?"

If we only consider "regular" packings with some kind of translational symmetry and assume all spheres are the same size then answers to this question are known for $1 \leq n \leq 8$.

For example here are the best such packings in dimensions $1, 2$ and 3:



There is a recursive process that explains all best packings for these dimensions.

A similar question asks for the maximum number of non-overlapping spheres that may touch the boundary of a single sphere. Again answers to this question are known for $1 \leq n \leq 8$. In fact the same recursive process mentioned above also solves this problem for these dimensions.

However it is quite remarkable that there is one other dimension in which this problem has been solved. We happen to know the answer in 24 dimensions!

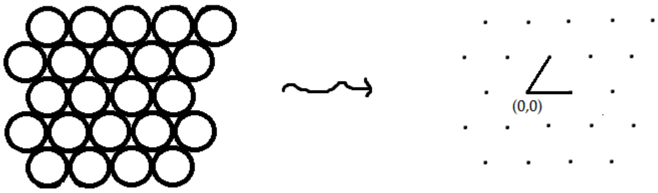
This is quite mysterious and the aim of this talk is to show how a solution is reached. In doing so we will take a step into the world of modular forms and will see a famous congruence of Ramanujan proved for very little extra effort.

Plan

1. Lattices
2. Modular Forms (for $\mathrm{SL}_2(\mathbb{Z})$)
3. Theta series
4. Two remarkable results

1 Lattices

Given a regular packing of spheres we can create a so-called “lattice”:



Definition 1.1. A lattice of rank r in $\mathbb{R}^n$ is a free $\mathbb{Z}$-module generated by r linearly independent vectors in $\mathbb{R}^n$. From now on we may assume that $r = n$ since these lattices model sphere packings in $\mathbb{R}^n$.

The above hexagonal lattice can be written as $\Lambda = \mathbb{Z}(1,0) \oplus \mathbb{Z}(\frac{1}{2}, \frac{\sqrt{3}}{2})$.

It should be noted that as subsets of $\mathbb{R}^n$ lattices inherit the usual inner product and norm (although a convention I will make is that $N(v) = ||v||^2$ is referred to as the norm).

Definition 1.2. An integral lattice Λ is one where $a.b \in \mathbb{Z}$ for all $a, b \in \Lambda$.

It is now clear that solving the maximum touching problem in $\mathbb{R}^n$ is equivalent to finding a lattice in $\mathbb{R}^n$ with the maximal number of “short” vectors, i.e. non-zero vectors of minimal norm.

A special lattice in 24-dimensions is the Leech lattice Λ_{24}. Explicitly it can be described as $\{v \in \mathbb{Z}^{25} \,|\, v.w = 0\}$ where $w = (3, 5, 7, \dots, 47, 51, -145)$. However this description is not important to us, the properties of the Leech lattice are more useful.

It is the unique lattice in $\mathbb{R}^{24}$ with the following properties:

1. $N(v) \in 2\mathbb{Z}$ for all $v \in \Lambda_{24}$ (Even lattice).
2. $N(v) \neq 2$ for every $v \in \Lambda_{24}$.
3. $\det(M) = 1$ where M is a generator matrix for Λ_{24} (Unimodular lattice)

2 Modular Forms

Modular forms are special functions on the upper half plane $\mathcal{H}$. They are associated with special subgroups of $\mathrm{SL}_2(\mathbb{Z})$ but in this talk we will only look at modular forms for the entire group $\mathrm{SL}_2(\mathbb{Z})$.

The group $\mathrm{SL}_2(\mathbb{Z})$ acts on $\mathcal{H}$ by Mobius transformations:

$$z \longmapsto \frac{az+b}{cz+d}$$

for each $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$.

We would like to study functions that behave nicely with respect to this action. We would like the following:

1. A complex valued function $f : \mathcal{H} \longrightarrow \mathbb{C}$ that is holomorphic.
2. The existence of an integer k such that $f(\frac{az+b}{cz+d}) = (cz+d)^k f(z)$ for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$.
3. For f to have no singularities at infinity. To explain this we note that since $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$, any function satisfying the first two properties must have period 1, i.e. $f(z+1) = f(z)$.

   In particular we have a complex Fourier series of the form:

   $$f(z) = \sum_{n=-\infty}^{\infty} a_n q^n$$

   where $q = e^{2\pi i z}$. The third condition means that we demand $a_n = 0$ for all $n < 0$.

Definition 2.1. Any function satisfying these properties is called a modular form of weight k (for $\mathrm{SL}_2(\mathbb{Z})$).

Before looking at some examples there are some nice facts about modular forms that are handy to know:

1. The weight k must be even and non-negative.
2. The space of weight k modular forms, denoted M_k, is a finite dimensional vector space with:

   $$\dim(M_k) = \begin{cases} \lfloor \frac{k}{12} \rfloor & \text{if } k \equiv 2 \bmod 12 \\ \lfloor \frac{k}{12} \rfloor + 1 & \text{otherwise} \end{cases}$$

Examples

1. $M_0 = \{\text{constant functions}\}$.

   However if we allow singularities at infinity then the j-function mentioned in Konstantinos' talk is an example (in fact this one generates all such functions as a $\mathbb{C}$-algebra.)

2. Eisenstein series (for even $k > 2$):

$$G_k(z) = \sum_{(m,n)\in\mathbb{Z}^2,(m,n)\neq(0,0)} \frac{1}{(mz+n)^k} \in M_k$$

The Fourier series is the following:

$$G_k(z) = 2\zeta(k) + \frac{(2\pi i)^k}{(k-1)!} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n$$

where ζ is the Riemann zeta function and $\sigma_{k-1}(n) = \sum_{d|n,d>0} d^{k-1}$.

This hints at the importance of modular forms in number theory. An analytic object somehow has number theoretical information encoded in its Fourier series.

One particular (scaled) Eisenstein series we will need is a weight 12 one, which has Fourier series:

$$E_{12}(z) = \frac{G_{12}(z)}{2\zeta(12)} = 1 + \frac{65530}{691} \sum_{n=1}^{\infty} \sigma_{11}(n) q^n.$$

3. We know that $\dim(M_{12}) = 2$ so how might we complete a basis?

The discriminant is the weight 12 modular form:

$$\Delta(z) = q \prod_{n=1}^{\infty} (1-q^n)^{24} = \sum_{n=1}^{\infty} \tau(n) q^n.$$

Clearly E_{12} and Δ are linearly independent so $M_{12} = \mathbb{C}E_{12} \oplus \mathbb{C}\Delta$.

Ramanujan studied the numbers $\tau(n)$ extensively and made many conjectures about them, some of which led to big advances in the theory of modular forms. One particular observation of his is the congruence:

$$\tau(n) \equiv \sigma_{11}(n) \bmod 691$$

This basically tells us that the two modular forms E_{12} and Δ are the "same" mod 691 despite being completely different in $\mathbb{C}$.

We will prove this congruence later.

3 Theta series

Given an integral lattice Λ we may define it's theta series:

$$\Theta_\Lambda(z) = \sum_{v\in\Lambda} q^{\frac{N(v)}{2}} = \sum_{m=0}^{\infty} N_m q^{\frac{m}{2}}$$

where N_m is the number of norm m vectors in Λ. The reason for the half in the power will become clear soon.

Examples

1. $\Theta_{\mathbb{Z}} = 1 + 2\sum_{m=1}^{\infty} q^{\frac{m^2}{2}}$.
2. $\Theta_{\mathbb{Z}(1,0)\oplus\mathbb{Z}(0,1)} = \sum_{m=0}^{\infty} r_2(n) q^{\frac{m}{2}}$ where $r_2(n)$ is the number of ways of writing n as a sum of two squares. Notice that $\Theta_{\mathbb{Z}(1,0)\oplus\mathbb{Z}(0,1)} = \Theta_{\mathbb{Z}}^2$.

Result If $\Lambda \subseteq \mathbb{R}^n$ is integral, even and unimodular then $\Theta_\Lambda \in M_{\frac{n}{2}}$ (recall the assumption that Λ has full rank).

4 Two remarkable results

Theorem

1. The maximum touching problem in $\mathbb{R}^{24}$ has solution 196560.
2. The Ramanujan congruence $\tau(m) \equiv \sigma_{11}(m) \bmod 691$ holds for all $m \geq 1$.

Proof We know that $\Theta_{\Lambda_{24}} \in M_{12} = \mathbb{C}E_{12} \oplus \mathbb{C}\Delta$, so that:

$$\Theta_{\Lambda_{24}} = \alpha E_{12} + \beta\Delta$$

for some $\alpha, \beta \in \mathbb{C}$. However we know that there are no norm 2 vectors in the Leech lattice so the theta series is of the form:

$$\Theta_{\Lambda_{24}} = 1 + 0q + \ldots$$

Comparing coefficients gives $\alpha = 1$ and $\beta = -\frac{65530}{691}$.

This now tells us the rest of the coefficients of the theta series. It tells us that for each m:

$$N_{2m} = \frac{65530}{691}(\sigma_{11}(m) - \tau(m))$$

In particular for $m = 2$ it tells us that there are $N_4 = \frac{65530}{691}(\sigma_{11}(2) - \tau(2)) =$ 196560 vectors of norm 4 in the Leech lattice. So the maximum touching problem in $\mathbb{R}^{24}$ has answer that is at least this number, but an upper bound of Odlyzko/Sloane confirms this as the optimum.

The congruence is easy to see now since we know beforehand that $N_{2m} \in \mathbb{Z}$. Thus 691 divides $\sigma_{11}(m) - \tau(m)$ for all $m \geq 1$.