Paramodular Eisenstein congruences for GSp_4.

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Outline of talk

Let $\Delta \in S_{12}(\mathrm{SL}_2(\mathbb{Z}))$ be the discriminant function. It is a normalized Hecke eigenform with Hecke eigenvalues $\tau(n)$ (the Ramanujan tau function).

Since $\mathrm{ord}_{691}\left(\frac{\zeta(12)}{\pi^{12}}\right) > 0$ we observe a congruence with the weight 12 Eisenstein series:

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More "Eisenstein congruences" predicted to occur. For example genus 2 Siegel modular forms:

Harder's conjecture

Let $j > 0, k \geq 3$ and let $f \in S_{j+2k-2}(\mathrm{SL}_2(\mathbb{Z}))$ be a normalized eigenform. Suppose $\mathrm{ord}_\lambda\left(\frac{L(f,j+k)}{\Omega}\right) > 0$ for some "large prime" λ (here Ω is a canonical period).

Then there exists an eigenform $F \in S_{j,k}(\mathrm{Sp}_4(\mathbb{Z}))$ such that for all primes p:

$$a_{F,p} \equiv a_{f,p} + p^{j+k-1} + p^{k-2} \bmod \Lambda$$

where $\Lambda | \lambda$ in $\mathbb{Q}_F\mathbb{Q}_f$.

Harder's conjecture - level p paramodular version

Let $j > 0, k \geq 3$ and let $f \in S^{new}_{j+2k-2}(\Gamma_0(p))$ be a normalized eigenform. Suppose $\mathrm{ord}_\lambda\left(\frac{L(f,j+k)}{\Omega}\right) > 0$ for some "large prime" λ (here Ω is a canonical period).

Then there exists an eigenform $F \in S^{new}_{j,k}(K(p))$ such that for all primes $q \neq p$:

$$a_{F,q} \equiv a_{f,q} + q^{j+k-1} + q^{k-2} \bmod \Lambda$$

where $\Lambda | \lambda$ in $\mathbb{Q}_F \mathbb{Q}_f$ and Λ doesn't divide p.

What is known?

Only one single instance of this congruence has been proved!
Evidence is quite hard to come by too...

- Level 1: Faber, Van der Geer for 1-dimensional spaces of forms.
- Level 2: Bergstrom et al, also for 1-dimensional spaces (method specific to level 2).
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Outline of talk

1. Harder's conjecture
2. Ibukiyama's correspondence
3. Computations

Question: How to get $a_{F,q}$ for $q \neq p$?

Ibukiyama's conjecture

There exists a Hecke-equivariant isomorphism:

$$S_{j,k}^{\text{new}}(K(p)) \cong \mathcal{A}_{j,k-3}^{\text{new}},$$

where $\mathcal{A}_{j,k-3}$ is a suitable space of genus 2 algebraic modular forms of level 1.

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What is $\mathcal{A}_{j,k-3}$? Be patient!

Let $D/\mathbb{Q}$ be a quaternion algebra with discriminant p and consider the reductive group $G/\mathbb{Q}$ whose F-rational points are:

$$G(F) = \{A \in M_2(D \otimes F) \mid A\overline{A}^T = \mu I, \mu \in (\mathbb{Q} \otimes F)^\times\}.$$

Facts

- G is an inner form of GSp_4.
- $G(\mathbb{R})/Z(G(\mathbb{R})) \cong \mathrm{USp}(4)/\{\pm I\}$ is compact.
- For $q \neq p$ we have $G(\mathbb{Q}_q) \cong \mathrm{GSp}_4(\mathbb{Q}_q)$.

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Let $V_{j,k-3}$ be the irrep of USp(4) with Young diagram parameters $(j+k-3, k-3)$. (A "weight").

Let $K_f = \mathrm{Stab}_{G(\mathbb{A}_f)}(L)$ where L is a special kind of rank 2 $\mathcal{O}$-lattice ("non-principal genus"). (A "level" structure).

$\mathcal{A}_{j,k-3}$ is the space of functions $f : G(\mathbb{A}_f) \to V_{j,k-3}$ satisfying

$$f(\gamma g k) = \gamma \cdot f(g)$$

for all $(\gamma, g, k) \in G(\mathbb{Q}) \times G(\mathbb{A}_f) \times K_f$

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Hecke operators

Each $u \in G(\mathbb{A}_f)$ defines a Hecke operator T_u via:

$$T_u(f)(g) = \sum_{i=1}^{r} f(gu_i),$$

where $K_f u K_f = \coprod_{i=1}^{r} u_i K_f$.

Ibukiyama predicts:

$$T_q \longleftrightarrow T_u$$

where u has identity component away from q and $u_q \mapsto \mathrm{diag}(1, 1, q, q)$ under a fixed isomorphism $G(\mathbb{Q}_q) \cong \mathrm{GSp}_4(\mathbb{Q}_q)$.

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Assume $p \in \{2, 3, 5, 7, 11\}$ for simplicity.

Fact

$$G(\mathbb{A}_f) = G(\mathbb{Q})K_f.$$

So every $f \in \mathcal{A}_{j,k-3}$ is determined by $f(\mathrm{id}) \in V_{j,k-3}$. There are further restrictions:

Lemma

The map $f \mapsto f(\mathrm{id})$ defines an isomorphism:

$$\mathcal{A}_{j,k-3} \cong V_{j,k-3}^{\Gamma},$$

where $\Gamma = G(\mathbb{Q}) \cap K_f$. This is a finite group due to compactness of $G(\mathbb{R})/Z(G(\mathbb{R}))$.

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How to compute Γ?

We can choose a $g \in G(\mathbb{Q})$ such that $L = \mathcal{O}^2 g$.

Let $G(\mathbb{Q})_1$ be the subgroup of similitude 1 matrices. Then:

$$\Gamma = G(\mathbb{Q}) \cap K_f = \mathrm{Stab}_{G(\mathbb{Q})}(L) = G(\mathbb{Q})_1 \cap g^{-1}\mathrm{GL}_2(\mathcal{O})g.$$

$$|\Gamma| = \frac{5760}{p^2 - 1}.$$

With a nice choice of g one can get an algorithm for finding the elements of this set by solving some norm equations.

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How to compute Hecke representatives for T_u?

By local considerations we find a similar description:

Let $G(\mathbb{Q})_q$ be the subgroup of similitude q matrices and $Y_q = G(\mathbb{Q})_q \cap g^{-1} M_2(\mathcal{O})^\times g$. Then:

$$K_f u K_f = \coprod_{[u_i] \in Y_q/\Gamma} u_i K_f.$$

$$\deg(T_u) = (p+1)(p^2+1).$$

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For example for $q = 2$ we get the following Hecke representatives (X_2 is the set of integral norm 2 quaternions):

$$\left\{ \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} \mid x, y \in X_2/\mathcal{O}^\times \right\} \cup \left\{ \begin{pmatrix} 1 & z_i \\ -\overline{z_i} & 1 \end{pmatrix} \mid [z_i] \in \mathcal{O}^\times/\{\pm 1\} \right\}.$$

Finally we can see a congruence! (well just T_2 today).

- Take $(p, j, k) = (3, 2, 8)$.
- $\dim(\mathcal{A}^{\text{new}}_{2,5}) = \dim(\mathcal{A}_{2,5}) = \dim(S^{\text{new}}_{2,8}(K(3))) = 1$.
- $S^{\text{new}}_{16}(\Gamma_0(3))$ contains an eigenform f with $a_{f,2} = -234$. Also the large prime 109 divides the numerator of the (normalized) L-value $\frac{L(f,10)}{\Omega}$.

So we expect a congruence:

$$a_{F,q} \equiv a_{f,q} + q^9 + q^6 \bmod 109,$$

where $F \in S^{\text{new}}_{2,8}(K(3))$ is the unique eigenform up to scaling.

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Running the algorithms to find Γ and the Hecke reps at $q = 2$ we can compute (using a trace formula):

$$\mathrm{tr}(T_u) = -312.$$

Voila!

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Thanks for listening!