MAS430 - Chapter 1 Solutions

Infinitude of primes

1. For any such factorisation we have $N \geq 2$ hence there exists a prime p such that $p \mid N$.

   However by assumption $p_1, p_2, ..., p_n$ are all of the primes, so that $p = p_i$ for some i.

   Now clearly $p_i \mid P = mn$ and so $p \mid m$ or $p \mid n$. Suppose $p_i \mid m$ (it doesn't matter which occurs).

   We now get our contradiction since:

   $$p_i \mid N - m = n$$

   so that $p_i^2 \mid P$ (which is false since P is square-free).

2. Note that a and $(a+1)$ are coprime since if $d \mid a$ and $d \mid (a+1)$ then $d \mid (a+1) - a = 1$.

   Also $a + 1 \geq 2$ hence there exists a prime $p \mid (a+1)$. By coprimality $p \nmid a$ and so $a(a+1)$ must have at least $r+1$ prime divisors.

   To prove the infinitude of primes we set up a recursion. Let $a_1 = 2$ and define:

   $$a_{n+1} = a_n(a_n + 1).$$

   By induction it is then clear that a_k has at least k prime divisors. This increases without bound as k increases, hence there must be infinitely many primes.

3. (a) It is a simple (although tedious) calculation to check the first claim. As for the second, note that $f(x) = x(x+1) + 41$ and from this it is obvious that $f(40)$ and $f(41)$ are both divisible by 41 and are bigger than 41, hence not prime.

   (b) Let $f(x) = a_n x^n + a_{n-1}x^{n-1} + ... + a_1 x + a_0$. Following the hint let $f(1) = p$ (this is prime by assumption).
   Consider $f(kp+1)$ for $k \in \mathbb{Z}$. Expanding gives:

   $$\begin{aligned} f(kp+1) &= a_n(kp+1)^n + a_{n-1}(kp+1)^{n-1} + ... + a_1(kp+1) + a_0 \\ &= pN + (a_n + a_{n-1} + ... + a_1 + a_0) \\ &= pN + f(1) \\ &= p(N+1) \end{aligned}$$

for some N.

However, by assumption $f(kp+1)$ is prime and $p \mid f(kp+1)$, hence $f(kp+1) = p$.

But now we have a contradiction since letting $g(x) = f(x) - p$ we see that:

$$g(kp+1) = f(kp+1) - p = p - p = 0$$

for **every** $k \in \mathbb{Z}$. Hence $g(x)$ has infinitely many roots which is a contradiction.

4. (a) Suppose there are k multiples of n^2 less than or equal to N. Then $n^2, 2n^2, 3n^2, ..., kn^2 \leq N$ but $(k+1)n^2 > N$.

   Thus:

   $$kn^2 \leq N < (k+1)n^2$$

   so that

   $$k \leq \frac{N}{n^2} < k+1,$$

   hence $\left\lfloor \frac{N}{n^2} \right\rfloor = k$, as required.

   (b) The quantity $N - A(N)$ measures how many integers less than or equal to N are **not** square-free, i.e. are divisible by at least one of $2^2, 3^2, ...$

   Clearly then by the first part:

   $$N - A(N) \leq \sum_{n=2}^{\infty} \left\lfloor \frac{N}{n^2} \right\rfloor \leq \sum_{n=2}^{\infty} \frac{N}{n^2}.$$

   (c) Using the previous inequality we see that:

   $$A(N) \geq N - \sum_{n=2}^{\infty} \frac{N}{n^2} = N\left(1 - \sum_{n=2}^{\infty} \frac{1}{n^2}\right) = N\left(1 - \left(\frac{\pi^2}{6} - 1\right)\right)$$
   $$= N\left(2 - \frac{\pi^2}{6}\right).$$

   Suppose there are finitely many primes $p_1, p_2, ..., p_m$. Then each square free integer would be of the form $p_1^{e_1} p_2^{e_2} ... p_m^{e_m}$ for $e_i \in \{0, 1\}$. There are 2^m such numbers so that $A(N) \leq 2^m$ for **any** N. In particular $A(N)$ is bounded for all N.

   However since $\left(2 - \frac{\pi^2}{6}\right) > 0$ we see from the above inequality that $A(N) \to \infty$ as $N \to \infty$. This is a contradiction.

Primes in arithmetic progressions

1. Suppose there are finitely many primes of the form $6k - 1$. Label these $p_1, p_2, ..., p_m$.

   Construct the number $N = 6p_1p_2...p_m - 1$.

   It is clear that $N \geq 2$ and so there exists a prime $p \mid N$.

Claim - N has a prime divisor of the form $6k-1$.

Suppose not. Then **all** prime divisors of N are of the form $6k+1$. But then since N is a product of numbers that are 1 mod 6 it must be that $N \equiv 1 \bmod 6$. This is a contradiction since by construction $N \equiv -1 \bmod 6$.

Thus there exists a prime $p \mid N$ such that p is of the form $6k-1$. But then $p = p_i$ for some i and so:

$$p_i \mid N - (6p_1p_2...p_m) = -1,$$

which gives a contradiction.

2. (a) Suppose the values $\Phi_q(1), \Phi_q(2), \Phi_q(3), ...$ have only a finite number of prime divisors. Label these $p_1, p_2, ..., p_m$.
   Then since $\Phi_q(p_1p_2...p_m) \geq 2$ there exists p_i such that $p_i \mid \Phi_q(p_1p_2...p_m)$. But:

   $$\begin{aligned}\Phi_q(p_1p_2...p_m) &= (p_1p_2...p_m)^{q-1} + (p_1p_2...p_m)^{q-2} ... + (p_1p_2...p_m) + 1 \\ &= p_iN + 1\end{aligned}$$

   for some $N \geq 1$. Thus $p_i \mid 1$, giving a contradiction.

   (b) We write $x^q - 1 = (x-1)\Phi_q(x)$ and then it is clear that if $p \mid \Phi_q(a)$ for some a then $p \mid a^q - 1$, so that $a^q \equiv 1 \bmod p$.

   By basic group theory it is now true that the order of $a \bmod p$ must divide q, so is either 1 or q.

   However it can't be 1. If it were then $a \equiv 1 \bmod p$ and so:

   $$\Phi_q(a) = a^{q-1} + a^{q-2} + ... + a + 1 \equiv 1 + 1 + ... + 1 + 1 = q \bmod p,$$

   giving $q \equiv 0 \bmod p$ (since $\Phi_q(a) \equiv 0 \bmod p$).

   But this is a contradiction since $q > p$ is prime.

   (c) We know by the previous part that the order of $a \bmod p$ is q. But the order of the group $(\mathbb{Z}/p\mathbb{Z})^\times$ is $p-1$.

   By Lagrange's theorem we must then have $q \mid (p-1)$ so that $p \equiv 1 \bmod q$.

   To prove the second claim note that by the first part there are infinitely many prime divisors among the values $\Phi_q(1), \Phi_q(2), \Phi_q(3), ...$, so there are certainly infinitely many that are bigger then q.

   But we have just proved that any such prime divisor must be 1 mod q. So there must be infinitely many primes congruent to 1 mod q.

3. Suppose L has length n. We know that L is coprime to 10^n (since L is not even and not divisible by 5).

   So by Dirichlet's theorem there are infinitely many primes of the form $10^nk + L$. Such primes have L as the last n digits.

Bertrand's postulate and Chebyshev's inequalities

1. (a) There are $\left\lfloor \frac{n}{p} \right\rfloor$ multiples of p less than n. Each of these contributes at least 1 to the power of p dividing $n!$. We have undercounted due to multiples of p^2.

   Any multiple of p^2 contributes at least 2 to the power of p dividing $n!$. There are $\left\lfloor \frac{n}{p^2} \right\rfloor$ of these. Thus $\left\lfloor \frac{n}{p} \right\rfloor + \left\lfloor \frac{n}{p^2} \right\rfloor$ accounts for the correct contibution of multiples of the form mp and mp^2 for $p \nmid m$. However we have undercounted due to multiples of p^3.

   Any multiple of p^3 contributes at least 3 to the power of p dividing $n!$. There are $\left\lfloor \frac{n}{p^3} \right\rfloor$ of these. Thus $\left\lfloor \frac{n}{p} \right\rfloor + \left\lfloor \frac{n}{p^2} \right\rfloor + \left\lfloor \frac{n}{p^3} \right\rfloor$ accounts for the correct contibution of multiples of the form mp, mp^2 and mp^3 for $p \nmid m$. However we have undercounted due to multiples of p^4.

   Continuing on with this process we see that:

$$\sum_{k=1}^{\infty} \left\lfloor \frac{n}{p^k} \right\rfloor$$

   should measure the correct contribution to the power for each multiple of p.

   This is not really an infinite sum. Notice that if $p^{s_p} \leq n < p^{s_p+1}$ then $\frac{n}{p^{s_p+1}} < 1$ so that the terms from this index onwards are all 0.

   (b) Clearly the exact power of p dividing $\frac{(2n)!}{n!n!}$ is going to be the exact power of p dividing $(2n)!$ minus double the exact power of p dividing $n!$ (to account for cancellation).

   By part (a) this is:

$$\sum_{k=1}^{\infty} \left(\left\lfloor \frac{2n}{p^k} \right\rfloor - 2 \left\lfloor \frac{n}{p^k} \right\rfloor \right)$$

   and since $p^{r_p} \leq 2n < p^{r_p+1}$ we see that $\frac{2n}{p^{r_p+1}} < 1$ hence all terms beyond $k = r_p$ turn out the be zero. Thus we may take r_p as the upper limit in the sum.

2. (a) Clearly the result is true for $n = 2$. Also for any even $n > 2$ we know that $n = 2m$ cannot be prime, hence:

$$\prod_{p \leq 2m} p = \prod_{p \leq 2m-1} p.$$

   If we knew the result for odd n then:

$$\prod_{p \leq 2m-1} p < 4^{2m-1} < 4^{2m},$$

   proving the result for even n.

   (b) Note that:

$$\binom{2m+1}{m} = \binom{2m+1}{m+1}.$$

Also:

$$\binom{2m+1}{m} + \binom{2m+1}{m+1} \leq (1+1)^{2m+1} = 2^{2m+1}$$

by the binomial expansion.

Thus:

$$\binom{2m+1}{m} \leq \frac{2^{2m+1}}{2} = 2^{2m} = 4^m.$$

(c) The result is certainly true for $n = 1, 2$ and given $n \geq 1$ we assume the result is true for all $1 \leq k \leq n-1$.

By part (a) we may assume that $n = 2m+1$ is odd. Then by the inductive hypothesis:

$$\prod_{p \leq 2m+1} p = \left(\prod_{p \leq m+1} p \right) \left(\prod_{m+1 < p \leq 2m+1} p \right) < 4^{m+1} \prod_{m+1 < p \leq 2m+1} p.$$

Finally note that each prime satisfying $m+1 < p \leq 2m+1$ must divide $\binom{2m+1}{m}$ (since p divides $(2m)!$ but not $m!$ or $(m+1)!$).

Thus by part (b):

$$\left(\prod_{m+1 < p \leq 2m+1} p \right) \leq \binom{2m+1}{m} \leq 4^m.$$

It now follows that:

$$\prod_{p \leq 2m+1} p < 4^{m+1} \cdot 4^m = 4^{2m+1}$$

and so by strong induction the result is true.

3. By Bertrand's postulate there is always a prime between 10^n and 2×10^n for $n = 1, 2, 3, \ldots$. Such primes begin with digit 1.

4. (a) Creating a common denominator of $n!$ we see that:

$$\sum_{k=1}^{n} \frac{1}{k} = \frac{\frac{n!}{1} + \frac{n!}{2} + \ldots + \frac{n!}{n-1} + \frac{n!}{n}}{n!}.$$

By Bertrand's postulate there exists a prime p such that $\frac{n}{2} < p < n$.

Now $p \mid \frac{n!}{m}$ for any $m \neq p$ (since p has not been cut out of the product).

Also since $2p > n$ the only number between 1 and n that is divisible by p is p itself. Hence $p \nmid \frac{n!}{p}$.

Thus p divides the denominator of the fraction but **not** the numerator. Hence the fraction cannot be a integer.

(b) By Bertrand's postulate there exists a prime p such that $\frac{n}{2} < p < n$.

If $n!$ were a square then $p^2 \mid n!$.

However, $2p > n$ and so $p^2 \nmid n!$

(Alternatively use the formula above to show that the exact power of p dividing $n!$ is p, not p^2).

Asymptotics and the prime number theorem

1. First the relation is reflexive since:

$$\lim_{x\to\infty}\frac{f(x)}{f(x)}=\lim_{x\to\infty}1=1$$

showing that $f\sim f$.

Next suppose that $f\sim g$. Then by algebra of limits:

$$\lim_{x\to\infty}\frac{g(x)}{f(x)}=\lim_{x\to\infty}\frac{1}{\frac{f(x)}{g(x)}}=\frac{1}{1}=1$$

thus $g\sim f$, giving the symmetric property.

Finally we show transitivity. Suppose $f\sim g$ and $g\sim h$. Then by algebra of limits:

$$\lim_{x\to\infty}\frac{f(x)}{h(x)}=\lim_{x\to\infty}\frac{f(x)}{g(x)}\frac{g(x)}{h(x)}=\lim_{x\to\infty}\frac{f(x)}{g(x)}\lim_{x\to\infty}\frac{g(x)}{h(x)}=1\cdot 1=1$$

Thus $f\sim h$.

2. (a)

$$\lim_{x\to\infty}\frac{x^3+4x+5}{x^3}=\lim_{x\to\infty}1+\frac{4}{x^2}+\frac{5}{x^3}=1.$$

(b) Let $P(x)=\alpha_n x^n+\alpha_{n-1}x^{n-1}+...+\alpha_1 x+\alpha_0$. Then:

$$\lim_{x\to\infty}\frac{P(x)}{\alpha_n x^n}=\lim_{x\to\infty}\left(1+\frac{\alpha_{n-1}}{\alpha_n x}+...+\frac{\alpha_1}{\alpha_n x^{n-1}}+\frac{\alpha_0}{\alpha_n x^n}\right)=1.$$

(c) For $x>0$:

$$\lim_{x\to\infty}\frac{\sqrt{x^2+A}}{x}=\lim_{x\to\infty}\sqrt{1+\frac{A}{x^2}}=1.$$

(d) For $x>-A$:

$$\lim_{x\to\infty}\frac{\frac{1}{x}}{\frac{1}{x+A}}=\lim_{x\to\infty}\frac{x+A}{x}=\lim_{x\to\infty}1+\frac{A}{x}=1.$$

3. (a) Since $f(x)\to\infty$ as $x\to\infty$ we know that $\frac{A}{f(x)}\to 0$.
Then

$$\lim_{x\to\infty}\frac{f(x)+A}{f(x)}=\lim_{x\to\infty}1+\frac{A}{f(x)}=1+\lim_{x\to\infty}\frac{A}{f(x)}=1.$$

(b) Suppose $A\le g(x)\le B$ for constants A,B. Then for all x:

$$\frac{A}{f(x)}\le\frac{g(x)}{f(x)}\le\frac{B}{f(x)}.$$

By the sandwich rule it now follows that $\frac{g(x)}{f(x)}\to 0$ as $x\to\infty$.
Then

$$\lim_{x\to\infty}\frac{f(x)+g(x)}{f(x)}=\lim_{x\to\infty}1+\frac{g(x)}{f(x)}=1+\lim_{x\to\infty}\frac{g(x)}{f(x)}=1.$$

(c) A general example is when $g(x) = f(x)$. Then:

$$\lim_{x\to\infty} \frac{f(x)+f(x)}{f(x)} = 2 \neq 1$$

so that $f(x) + f(x) \nsim f(x)$.

4. By algebra of limits:

$$\lim_{x\to\infty} \frac{f(x)+g(x)}{2h(x)} = \lim_{x\to\infty} \frac{f(x)}{2h(x)} + \lim_{x\to\infty} \frac{g(x)}{2h(x)} = \frac{1}{2} + \frac{1}{2} = 1.$$

The second claim is clear since:

$$\lim_{x\to\infty} \frac{Af(x)}{Ag(x)} = \lim_{x\to\infty} \frac{f(x)}{g(x)} = 1.$$

5. We partition the primes less than x into classes mod m:

$$\pi(x) = \sum_{p|m} 1 + \sum_{\bar{b}\in(\mathbb{Z}/m\mathbb{Z})^\times} \pi_{m,b}(x).$$

Now $\sum_{p|m} 1$ is constant hence by Exercise 3 we have:

$$\pi(x) \sim \sum_{\bar{b}\in(\mathbb{Z}/m\mathbb{Z})^\times} \pi_{m,b}(x).$$

Since we are assuming the result that $\pi_{m,b}(x) \sim \pi_{m,c}(x)$ for any b, c coprime to m we can use Exercise 4 to see that:

$$\pi(x) \sim \pi_{m,a}(x) + \pi_{m,a}(x) + ... + \pi_{m,a}(x) = \phi(m)\pi_{m,a}(x).$$

By transitivity of $\sim$ along with the PNT we now see that:

$$\phi(m)\pi_{m,a}(x) \sim \frac{x}{\ln(x)},$$

so that $\pi_{m,a}(x) \sim \frac{x}{\phi(m)\ln(x)}$.

6. Using integration by parts:

$$\int_2^x \frac{dt}{\ln(t)} = \frac{x}{\ln(x)} - \frac{2}{\ln(2)} + \int_2^x \frac{dt}{(\ln(t))^2}.$$

Thus:

$$\frac{\int_2^x \frac{dt}{\ln(t)}}{x/\ln(x)} = 1 - \frac{2\ln(x)}{x\ln(2)} + \frac{\ln(x)}{x}\int_2^x \frac{dt}{(\ln(t))^2}.$$

Now $\lim_{x\to\infty} \frac{\ln(x)}{x} = \lim_{x\to\infty} \frac{1}{x} = 0$ by L'Hopital's rule. Thus the second term of the above tends to 0 as $x \to \infty$. We wish to show the third term also tends to 0 as $x \to \infty$ and then we will be done since the limit of the entire RHS will be 1.

Unfortunately we cannot yet say that the third term tends to 0 as $x \to \infty$ (in fact the integral $\int_2^\infty \frac{dt}{(\ln(t))^2}$ doesn't exist so algebra of limits does **not** work here).

Clever trick alert: Instead let's split the integral up and use the fact that $f(t) = \frac{1}{(\ln(t))^2}$ is strictly decreasing for $t > 1$.

$$\begin{aligned}
\frac{\ln(x)}{x}\int_2^x \frac{dt}{(\ln(t))^2} &= \frac{\ln(x)}{x}\left(\int_2^{\sqrt{x}} \frac{dt}{(\ln(t))^2} + \int_{\sqrt{x}}^x \frac{dt}{(\ln(t))^2}\right) \\
&\leq \frac{\ln(x)}{x}\left(\frac{\sqrt{x}}{(\ln(2))^2} + \frac{x}{(\ln(\sqrt{x}))^2}\right) \\
&= \frac{\ln(x)}{\sqrt{x}(\ln(2))^2} + \frac{\ln(x)}{(\ln(\sqrt{x}))^2} \\
&= \frac{2\ln(\sqrt{x})}{(\ln(2))^2\sqrt{x}} + \frac{4}{\ln(x)}.
\end{aligned}$$

Thus:

$$0 \leq \frac{\ln(x)}{x}\int_2^x \frac{dt}{(\ln(t))^2} \leq \frac{2\ln(\sqrt{x})}{(\ln(2))^2\sqrt{x}} + \frac{4}{\ln(x)}.$$

The RHS clearly tends to 0 as $x \to \infty$ hence by the sandwich rule $\frac{\ln(x)}{x}\int_2^x \frac{dt}{(\ln(t))^2} \to 0$ as required.

7. Since $p_n \sim n\ln(n)$ we see that:

$$\begin{aligned}
\lim_{n\to\infty} \frac{p_n}{p_{n+1}} &= \lim_{n\to\infty} \frac{p_n}{n\ln(n)} \frac{n\ln(n)}{(n+1)\ln(n+1)} \frac{(n+1)\ln(n+1)}{p_{n+1}} \\
&= \lim_{n\to\infty} \frac{n\ln(n)}{(n+1)\ln(n+1)}.
\end{aligned}$$

This limit is 1 since:

$$\lim_{n\to\infty} \frac{n}{n+1} = \lim_{n\to\infty} \frac{1}{1+\frac{1}{n}} = 1$$

and by L'Hopital's rule:

$$\lim_{n\to\infty} \frac{\ln(n)}{\ln(n+1)} = \lim_{n\to\infty} \frac{\frac{1}{n}}{\frac{1}{n+1}} = \lim_{n\to\infty} \frac{n+1}{n} = \lim_{n\to\infty} 1 + \frac{1}{n} = 1.$$

8. (a) By PNT:

$$\begin{aligned}
\lim_{x\to\infty} \frac{\pi(bx)}{\pi(ax)} &= \lim_{x\to\infty} \frac{\pi(bx)}{bx/\ln(bx)} \frac{bx/\ln(bx)}{ax/\ln(ax)} \frac{ax/\ln(ax)}{\pi(ax)} \\
&= \frac{b}{a}\lim_{x\to\infty} \frac{\ln(ax)}{\ln(bx)} \\
&= \frac{b}{a}\lim_{x\to\infty} \frac{\frac{a}{ax}}{\frac{b}{bx}} \\
&= \frac{b}{a}
\end{aligned}$$

where the second to last equality is by L'Hopital's rule.

(b) Since $\frac{b}{a} > 1$ we may choose $\delta > 0$ such that $\frac{b}{a} > 1 + \delta$.

Now $\frac{\pi(bx)}{\pi(ax)} \to \frac{b}{a}$ as $x \to \infty$ so there exists $C > 0$ such that:

$$\frac{\pi(bx)}{\pi(ax)} > 1 + \delta,$$

for $x > C$.

However since $\pi(ax) \to \infty$ as $x \to \infty$ we see that there exists $D > 0$ such that $\delta\pi(ax) > 1$ for $x > D$.

Then for $x > \max\{C, D\}$ we see that $\pi(bx) - \pi(ax) > \delta\pi(ax) > 1$ so that there exists at least two primes between ax and bx (including bx). For such x there definitely exists at least one prime between ax and bx.

(c) We know that there exists $A > 0$ such that for $x > A$ there is a prime between Lx and $(L + 1)x$.

For any of the infinitely many n such that $n > \frac{\ln(A)}{\ln(10)}$ we have that $10^n > A$ and so there is a prime between $L \times 10^n$ and $(L + 1) \times 10^n$. Each such prime begins with the string L.

(d) There exists $B > 0$ such that for $x > B$ there is a prime between ax and bx. Since there are infinitely many primes we may choose one such that $q > B$.

Then we know that there is a prime p between aq and bq. Thus $aq < p < bq$ which rearranges to give $a < \frac{p}{q} < b$.

MAS430 - Chapter 2 Solutions

Arithmetic functions

1. (a) For $n = 1$ we note that $(u \star \mu)(1) = u(1)\mu(1) = 1 = I(1)$.
For $n \geq 2$ with prime factorisation $n = p_1^{a_1} p_2^{a_2} ... p_k^{a_k}$ note that:

$$\begin{aligned}(\mu \star u)(n) &= \sum_{d|n} \mu(d) u\left(\frac{n}{d}\right) \\ &= u(1) - \left(\sum_{i=1}^{k} u(p_i)\right) + \left(\sum_{i<j}\sum_{j=1}^{k} u(p_i p_j)\right) - ... + (-1)^k u(p_1 p_2 ... p_k) \\ &= 1 - \binom{k}{1} + \binom{k}{2} - ... + (-1)^k \binom{k}{k} \\ &= (1-1)^k \\ &= 0.\end{aligned}$$

When $n = 10$ we have $I(10) = 0$. The RHS is:

$$\mu(1)u(10) + \mu(2)u(5) + \mu(5)u(2) + \mu(10)u(1) = 1 - 1 - 1 + 1 = 0.$$

(b) For $n = 1$ we note that $(\Lambda \star u)(1) = \Lambda(1)u(1) = 0 = \ln(1)$.
For $n \geq 2$ with prime factorisation $n = p_1^{a_1} p_2^{a_2} ... p_k^{a_k}$ note that:

$$\begin{aligned}\sum_{d|n} \Lambda(d) u\left(\frac{n}{d}\right) &= \sum_{m=1}^{a_1} \Lambda(p_1^m) + \sum_{m=1}^{a_2} \Lambda(p_2^m) + ... + \sum_{m=1}^{a_k} \Lambda(p_k^m) \\ &= a_1 \ln(p_1) + a_2 \ln(p_2) + ... + a_k \ln(p_k) \\ &= \ln(p_1^{a_1} p_2^{a_2} ... p_k^{a_k}) \\ &= \ln(n).\end{aligned}$$

When $n = 10$ we have $\ln(10) = \ln(2) + \ln(5)$. The RHS is:

$$\Lambda(1)u(10) + \Lambda(2)u(5) + \Lambda(5)u(2) + \Lambda(10)u(1) = \ln(2) + \ln(5).$$

(c) We partition the numbers $1, 2, 3, ..., n$ according to their highest common factor with n. For $d \mid n$ let $S_d = \{1 \leq k \leq n \,|\, \mathrm{hcf}(k, n) = d\}$. Then $n = \sum_{d|n} |S_d|$.

Now $|S_d| = \phi\left(\frac{n}{d}\right)$ since $\text{hcf}(k,n) = d$ if and only if $\text{hcf}\left(\frac{k}{d}, \frac{n}{d}\right) = 1$ (and so there are $\phi\left(\frac{n}{d}\right)$ possibilities for $\frac{k}{d}$ by definition of ϕ).
Thus $n = \sum_{d|n} \phi\left(\frac{n}{d}\right) = \sum_{d|n} \phi(d)$ as required.
For $n = 10$ the RHS is:

$$\phi(1)u(10) + \phi(2)u(5) + \phi(5)u(2) + \phi(10)u(1) = 1 + 1 + 4 + 4 = 10.$$

2. (a) Let f, g, h be such functions.
It is clear by definition of Dirichlet convolution that $(f \star g)$ is another arithmetic function. So closure is apparent.
If we rewrite the definition of Dirichlet convolution as:

$$(f \star g)(n) = \sum_{ab=n} f(a)g(b)$$

then associativity is clear:

$$\begin{aligned}(f \star (g \star h))(n) &= \sum_{ab=n} f(a)(g \star h)(b) \\ &= \sum_{abc=n} f(a)g(b)h(c) \\ &= \sum_{ab=n} (f \star g)(a)h(b) \\ &= ((f \star g) \star h)(n).\end{aligned}$$

We know that $I \star f = f \star I = f$ and $I(1) = 1 \neq 0$ and so we have an identity element.
Finally since $f(1) \neq 0$ we are guaranteed a Dirichlet inverse.

(b) Let f, g be such functions. Then since $(f \star g)(1) = f(1)g(1) = 1$ we see that this subset is closed under Dirichlet convolution.
Clearly the identity belongs to this subset since $I(1) = 1$ by definition.
Finally recall that $f^{-1}(1) = \frac{1}{f(1)} = 1$ so this subset is closed under inverses.
So this subset is in fact a subgroup.

3. In any group the identity is the only element of order 1, which here is I. Suppose f satisfies $f \star f = I$. We show that the only solutions are $f = \pm I$ (it is clear that these **are** solutions).

Now $(f \star f)(1) = I(1) = 1$ gives $f(1) = \pm 1$.

Let p be a prime. Then $(f \star f)(p) = I(p) = 0$ implies $f(1)f(p) + f(p)f(1) = 0$ so that $2f(1)f(p) = 0$. But $f(1) \neq 0$ hence $f(p) = 0$.

By induction it then follows that $f(p^k) = 0$ for each $k \geq 1$. But f is multiplicative so we must have $f(n) = 0$ for all $n \geq 2$. Thus $f = \pm I$.

4. We have two cases. If $f(1) \neq 0$ then the Dirichlet inverse exists and so $f \star f = f$ implies by cancelation that $f = I$.

If $f(1) = 0$ then we must show that $f \star f = f$ implies $f = \mathbf{0}$. We use strong induction to show that $f(n) = 0$ for all n. We are assuming the base case holds already.

Suppose we know that $f(k) = 0$ for $k = 1, 2, ..., n-1$. Then:

$$f(n) = (f \star f)(n) = \sum_{d|n} f(d) f\left(\frac{n}{d}\right)$$
$$= 0$$

since for each $d \mid n$ we have $d < n$ or $\frac{n}{d} < n$ (or both).

5. We know that $\sigma = N \star u$ and from Exercise 1 we know that $N = \phi \star u$, thus $\sigma = (\phi \star u) \star u = \phi \star (u \star u) = \phi \star \sigma_0$.

   For $n = 10$ we have that $\sigma(10) = 1 + 2 + 5 + 10 = 18$.

   The RHS is:

$$\begin{aligned}\phi(1)\sigma_0(10) + \phi(2)\sigma_0(5) + \phi(5)\sigma_0(2) + \phi(10)\sigma_0(1) \\ = \sigma_0(10) + \sigma_0(5) + 4\sigma_0(2) + 4\sigma_0(1) \\ = 4 + 2 + 8 + 4 \\ = 18.\end{aligned}$$

6. (a) Note that $\phi(1) = 1$ and if m, n are coprime then by the Chinese remainder theorem:

$$(\mathbb{Z}/mn\mathbb{Z})^{\times} \cong (\mathbb{Z}/m\mathbb{Z})^{\times} \times (\mathbb{Z}/n\mathbb{Z})^{\times},$$

   so that $\phi(mn) = \phi(m)\phi(n)$. (There are more elementary proofs of this result, try to find one).

   Since $\sigma_\alpha = N_\alpha \star u$ and N_α, u are multiplicative it follows that σ_α is multiplicative. (Again there are elementary proofs of this fact which you may discover. Hint: since m, n are coprime the divisors of mn are uniquely written in the form ab for $a \mid m$ and $b \mid n$).

   Finally we observe that $\mu(1) = 1$ so it remains to check that $\mu(mn) = \mu(m)\mu(n)$ for coprime $m, n \geq 2$. Write $m = p_1^{a_1} p_2^{a_2} ... p_s^{a_s}$ and $n = q_1^{b_1} q_2^{b_2} ... q_t^{b_t}$ with $p_i \neq q_j$ for any i, j.
   If either a_k or b_l is greater than 1 then $\mu(mn) = 0 = \mu(m)\mu(n)$.
   Now suppose that $a_k = b_l = 1$ for all k, l. Then $\mu(m) = (-1)^s$ and $\mu(n) = (-1)^t$ so that $\mu(mn) = (-1)^{st} = (-1)^s(-1)^t = \mu(m)\mu(n)$ (note how we used that fact that $p_i \neq q_j$ here, otherwise mn would have a square prime divisor giving $\mu(mn) = 0$).

   (b) Since σ_α is multiplicative we know that:

$$\sigma_\alpha(n) = \prod_{i=1}^{m} \sigma_\alpha(p_i^{k_i}).$$

   Now the only positive divisors of $p_i^{k_i}$ are $1, p_i, p_i^2, ..., p_i^{k_i}$ hence:

$$\sigma_\alpha(p_i^{k_i}) = \sum_{j=0}^{k_i} p_i^{j\alpha} = \frac{p_i^{(k_i+1)\alpha} - 1}{p_i^\alpha - 1}.$$

(c) It is clear that any positive divisor of n has the form $p_1^{a_1}p_2^{a_2}...p_m^{a_m}$ for $a_i \in \{0, 1, 2, ..., k_i\}$.
Since there are $(1+k_i)$ choices for each a_i and the a_i's are independent to each other we see that:

$$\sigma_0(n) = \prod_{i=1}^{m}(1 + k_i).$$

Note that we cannot plug in $\alpha = 0$ into the above formula. However if we stop short one step in the previous solution and start with:

$$\sigma_\alpha(p_i^{k_i}) = \sum_{j=0}^{k_i} p_i^{j\alpha}$$

then we observe that for $\alpha = 0$:

$$\sigma_0(p_i^{k_i}) = \sum_{j=0}^{k_i} 1 = (1 + k_i).$$

Thus:

$$\sigma_0(n) = \prod_{i=1}^{m}(1 + k_i)$$

as required.

7. If f is completely multiplicative then:

$$\begin{aligned}(f \star \mu f)(n) = (\mu f \star f)(n) &= \sum_{d|n} \mu(d)f(d)f\left(\frac{n}{d}\right)\\ &= \sum_{d|n} \mu(d)f(n)\\ &= f(n)(\mu \star u)(n) = I(n).\end{aligned}$$

The converse is true! To see this assume that $f^{-1} = \mu f$. We already know that f is multiplicative so in order to prove that f is completely multiplicative we need only prove that $f(p^k) = f(p)^k$ for all primes p and $k \geq 1$.
Since $f^{-1} = \mu f$ we see that $(\mu f \star f)(p^k) = I(p^k) = 0$. Thus:

$$\sum_{d|p^k} \mu(d)f(d)f\left(\frac{p^k}{d}\right) = 0.$$

However this simplifies to $f(1)f(p^k) - f(p)f(p^{k-1}) = 0$ so that $f(p^k) = f(p)f(p^{k-1})$. Repeated application of this identity gives:

$$f(p^k) = f(p)f(p^{k-1}) = f(p)^2 f(p^{k-2}) = ... = f(p)^k.$$

(Rigorously an inductive proof is best.)

8. We know that $\sigma = N \star u$. Thus by Möbius inversion we see that $N = \sigma \star \mu$. Thus for all $n \geq 1$ we have:

$$n = N(n) = (\sigma \star \mu)(n) = \sum_{d|n} \sigma(d) \mu\left(\frac{n}{d}\right).$$

For $n = 20$ the RHS is:

$$\begin{aligned}
\sigma(1)\mu(20) + \sigma(2)\mu(10) + \sigma(4)\mu(5) + \sigma(5)\mu(4) + \sigma(10)\mu(2) + \sigma(20)\mu(1) \\
&= \sigma(2) - \sigma(4) - \sigma(10) + \sigma(20) \\
&= 3 - 7 - 18 + 42 \\
&= 45 - 25 \\
&= 20.
\end{aligned}$$

9. By Exercise 1 we have $\ln = \Lambda \star u$ so by Möbius inversion we have $\Lambda = \ln \star \mu$. Thus for all $n \geq 1$ we have:

$$\Lambda(n) = (\ln \star \mu)(n) = \sum_{d|n} \ln(d) \mu\left(\frac{n}{d}\right).$$

Now $\Lambda(20) = 0$ whereas the RHS is:

$$\begin{aligned}
\ln(1)\mu(20) + \ln(2)\mu(10) + \ln(4)\mu(5) + \ln(5)\mu(4) + \ln(10)\mu(2) + \ln(20)\mu(1) \\
&= \ln(2) - \ln(4) - \ln(10) + \ln(20) \\
&= \ln(40) - \ln(40) \\
&= 0
\end{aligned}$$

Also $\Lambda(81) = \Lambda(3^4) = \ln(3)$ whereas the RHS is:

$$\begin{aligned}
\ln(1)\mu(81) + \ln(3)\mu(27) + \ln(9)\mu(9) + \ln(27)\mu(3) + \ln(81)\mu(1) \\
&= \ln(81) - \ln(27) = \ln(3)
\end{aligned}$$

Dirichlet series and Euler products

1.
$$D(s, N_\alpha) = \sum_{n=1}^{\infty} \frac{n^\alpha}{n^s} = \sum_{n=1}^{\infty} \frac{1}{n^{s-\alpha}} = \zeta(s - \alpha).$$

Since $\sigma = N \star u$ we know that $D(s, \sigma) = D(s, N)D(s, u) = \zeta(s - 1)\zeta(s)$.

2. (a) Since $\sigma_\alpha = N_\alpha \star u$ we know that $D(s, \sigma_\alpha) = D(s, N_\alpha)D(s, u) = \zeta(s - \alpha)\zeta(s)$.
   (b) Setting $\alpha = 0$ then $D(s, \sigma_0) = \zeta(s)^2$.
   (c) By Exercise 5 we know $\sigma = \phi \star \sigma_0$ we see that $D(s, \phi)D(s, \sigma_0) = D(s, \sigma)$ so $D(s, \phi)\zeta(s)^2 = \zeta(s - 1)\zeta(s)$. Thus $D(s, \phi) = \frac{\zeta(s-1)\zeta(s)}{\zeta(s)^2} = \frac{\zeta(s-1)}{\zeta(s)}$.
   If we instead use the convolution $N = \phi \star u$ then $D(s, N) = D(s, \phi)D(s, u)$. Thus $\zeta(s - 1) = D(s, \phi)\zeta(s)$, giving the same identity.

3. Since μ is multiplicative then it has an Euler product of the form:

$$D(s,\mu)=\prod_p\left(1+\frac{\mu(p)}{p^s}+\frac{\mu(p^2)}{p^{2s}}+...\right)=\prod_p\left(1-\frac{1}{p^s}\right).$$

This is clearly the reciprocal of the Euler product for $\zeta(s)$ hence:

$$D(s,\mu)=\frac{1}{\zeta(s)}.$$

4. By Exercise 7 we know that $f^{-1}=\mu f$. Hence $D(s,f)D(s,\mu f)=1$ giving $D(s,f)=\frac{1}{D(s,\mu f)}$.

   Now μf is multiplicative (but not completely multiplicative) hence $D(s,\mu f)$ has an Euler product of the form:

$$D(s,\mu f)=\prod_p\left(1+\frac{(\mu f)(p)}{p^s}+\frac{(\mu f)(p^2)}{p^{2s}}+...\right)=\prod_p\left(1-\frac{f(p)}{p^s}\right).$$

   Thus:

$$D(s,f)=\prod_p\left(1-\frac{f(p)}{p^s}\right)^{-1}.$$

5. (a) Let $m=p_1^{a_1}p_2^{a_2}...p_k^{a_k}$ and $n=q_1^{b_1}q_2^{b_2}...q_j^{b_j}$ be coprime. Then m,n do not share any prime factors so that none of the p's are equal to any of the q's.

   Thus:

$$P(mn)=P(p_1^{a_1}p_2^{a_2}...p_k^{a_k}q_1^{b_1}q_2^{b_2}...q_j^{b_j})=k+j=P(m)+P(n).$$

   Now:

$$2^{P(mn)}=2^{P(m)+P(n)}=2^{P(m)}2^{P(n)},$$

   thus we have multiplicativity.

(b) Since $2^{P(n)}$ is multiplicative we have an Euler product of the form:

$$\begin{aligned}
D(s,2^{P(n)}) &= \prod_p \left(1+\frac{2^{P(p)}}{p^s}+\frac{2^{P(p^2)}}{p^{2s}}+...\right) \\
&= \prod_p \left(1+\frac{2}{p^s}+\frac{2}{p^{2s}}+...\right) \\
&= \prod_p \left(\left(\sum_{n=0}^{\infty}\frac{1}{p^{ns}}\right)+\left(\sum_{m=1}^{\infty}\frac{1}{p^{ms}}\right)\right) \\
&= \prod_p \left(1+\frac{1}{p^s}\right)\left(\sum_{n=0}^{\infty}\frac{1}{p^{ns}}\right) \\
&= \prod_p \left(1+p^{-s}\right)\left(\frac{1}{1-p^{-s}}\right) \\
&= \prod_p \frac{1-p^{2s}}{(1-p^{-s})^2} \\
&= \prod_p \frac{(1-p^{-s})^{-2}}{(1-p^{-2s})^{-1}} \\
&= \frac{\zeta(s)^2}{\zeta(2s)}
\end{aligned}$$

(c) Note that the number of factorisations $n=ab$ where a,b are coprime is equal to the number of ways of partitioning the primes dividing k into two sets. This equals $2^{P(n)}$ since there are $P(n)$ such primes and a choice of two sets to place each in.
Thus:

$$\sum_{\text{hcf}(a,b)=1}\frac{1}{a^s b^s}=\sum_{n=1}^{\infty}\frac{2^{P(n)}}{n^s}=D(s,2^{P(n)}).$$

It is then clear that the sum in question is $D(2,2^{P(n)})=\frac{\zeta(2)^2}{\zeta(4)}$.

6. (a) We know that:

$$1+x+x^2+...=\frac{1}{1-x}.$$

Differentiating both sides of this gives:

$$1+2x+3x^2+...=\frac{1}{(1-x)^2}.$$

Multiplying by x gives:

$$x+2x^2+3x^3+...=\frac{x}{(1-x)^2}.$$

Then finally differentiating again gives:

$$1+4x+9x^2+16x^3+...=\frac{d}{dx}\left(\frac{x}{(1-x)^2}\right)=\frac{1-x^2}{(1-x)^4}.$$

(b) σ_0 is multiplicative hence so is σ_0^2. Thus we have an Euler product of the form:

$$\begin{aligned} D(s,\sigma_0^2) &= \prod_p \left(1 + \frac{\sigma_0^2(p)}{p^s} + \frac{\sigma_0^2(p^2)}{p^{2s}} + ...\right) \\ &= \prod_p \left(1 + \frac{2^2}{p^s} + \frac{3^2}{p^{2s}} + ...\right) \\ &= \prod_p \left(\frac{1-(p^{-s})^2}{(1-p^{-s})^4}\right) \\ &= \frac{\zeta(s)^4}{\zeta(2s)} \end{aligned}$$

(Technically we should have checked convergence of the identity in part (a) before plugging in $x = p^{-s}$...but we wont bother since we only wanted a formal identity)

7. (a) As a formal identity we have $S_k(x) = \frac{1-x^{k+1}}{1-x}$.
Thus:

$$\begin{aligned} \sum_{n=0}^{\infty} S_n(\alpha)S_n(\beta)x^n &= \sum_{n=0}^{\infty} \frac{(1-\alpha^{n+1})(1-\beta^{n+1})}{(1-\alpha)(1-\beta)}x^n \\ &= \sum_{n=0}^{\infty} \frac{1-\alpha^{n+1}-\beta^{n+1}+(\alpha\beta)^{n+1}}{(1-\alpha)(1-\beta)}x^n \\ &= \frac{1}{(1-\alpha)(1-\beta)}\sum_{n=0}^{\infty}(x^n - \alpha(\alpha x)^n - \beta(\beta x)^n + \alpha\beta(\alpha\beta x)^n) \\ &= \frac{1}{(1-\alpha)(1-\beta)}\left(\frac{1}{1-x} - \frac{\alpha}{1-\alpha x} - \frac{\beta}{1-\beta x} + \frac{\alpha\beta}{1-\alpha\beta x}\right) \\ &= ... \\ &= \frac{(1-\alpha)(1-\beta)(1-\alpha\beta x)}{(1-\alpha)(1-\beta)(1-x)(1-\alpha x)(1-\beta x)(1-\alpha\beta x)} \\ &= \frac{(1-\alpha\beta x^2)}{(1-x)(1-\alpha x)(1-\beta x)(1-\alpha\beta x)} \end{aligned}$$

(b) We know that σ_a and σ_b are multiplicative so $\sigma_a\sigma_b$ is too. Note that for prime p and $k \geq 1$:

$$\sigma_a(p^k) = 1 + p^a + p^{2a} + ... + p^{ka} = S_k(p^a)$$

so $(\sigma_a\sigma_b)(p^k) = S_k(p^a)S_k(p^b)$

The Euler product for $D(s,\sigma_a\sigma_b)$ is of the form:

$$
\begin{aligned}
D(s,\sigma_a\sigma_b) &= \prod_p \left(1+\frac{(\sigma_a\sigma_b)(p)}{p^s}+\frac{(\sigma_a\sigma_b)(p^2)}{p^{2s}}+...\right)\\
&= \prod_p \left(\sum_{n=0}^{\infty}\frac{S_n(p^a)S_n(p^b)}{p^{ns}}\right)\\
&= \prod_p \left(\frac{1-p^{a+b-2s}}{(1-p^{-s})(1-p^{a-s})(1-p^{b-s})(1-p^{a+b-s})}\right)\\
&= \prod_p \frac{[(1-p^{-s})(1-p^{-(s-a)})(1-p^{-(s-b)})(1-p^{-(s-a-b)})]^{-1}}{(1-p^{-(2s-a-b)})^{-1}}\\
&= \frac{\zeta(s)\zeta(s-a)\zeta(s-b)\zeta(s-a-b)}{\zeta(2s-a-b)}
\end{aligned}
$$

8. Since we want a formal identity we will not care too much about convergence or validity of taking logs in this question (such issues can be tackled later in the course).

   First note that since Λ is only non-zero for prime powers:

$$D(s,\Lambda)=\sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^s}=\sum_p\left(\frac{\ln(p)}{p^s}+\frac{\ln(p)}{p^{2s}}+\frac{\ln(p)}{p^{3s}}...\right).$$

   Now taking logs of the Euler product for $\zeta(s)$ we see that:

$$\ln(\zeta(s))=-\sum_p \ln\left(1-\frac{1}{p^s}\right)=\sum_p\sum_{n\geq 1}\frac{1}{np^{ns}}.$$

   (The final equality uses the Taylor expansion $\ln(1-x)=-x-\frac{x^2}{2}-\frac{x^3}{3}-...$). Differentiating both sides with respect to s and assuming we may interchange the derivative and the sum:

$$\frac{\zeta'(s)}{\zeta(s)}=\sum_p\sum_{n\geq 1}\frac{d}{ds}\left(\frac{1}{np^{ns}}\right).$$

   However by writing $p^{-ns}=e^{(-ns)\ln(p)}$ we find:

$$\frac{d}{ds}\left(\frac{1}{np^{ns}}\right)=-\frac{\ln(p)}{p^{ns}}.$$

   Thus:

$$\frac{\zeta'(s)}{\zeta(s)}=-\sum_p\sum_{n\geq 1}\frac{\ln(p)}{p^{ns}}=-D(s,\Lambda).$$

9. (a) Clearly not, for example $f(p^2)=f(p)f(p)-p^{k-1}f(1)=f(p)f(p)-p^{k-1}\neq f(p)f(p)$.

(b) Since f is multiplicative we have an Euler product of the form:

$$D(s,f) = \prod_p \left(1 + \frac{f(p)}{p^s} + \frac{f(p^2)}{p^{2s}} + ...\right).$$

It remains to show that:

$$\left(\sum_{n=0}^{\infty} \frac{f(p^n)}{p^{ns}}\right) = \frac{1}{1 - f(p)p^{-s} + p^{k-1-2s}}.$$

But this follows from the following:

$$\begin{aligned}
&(1 - f(p)p^{-s} + p^{k-1-2s}) \left(\sum_{n=0}^{\infty} \frac{f(p^n)}{p^{ns}}\right) \\
&= \left(\sum_{n=0}^{\infty} \frac{f(p^n)}{p^{ns}}\right) - \left(\sum_{n=0}^{\infty} \frac{f(p)f(p^n)}{p^{(n+1)s}}\right) + \left(\sum_{n=0}^{\infty} \frac{p^{k-1}f(p^n)}{p^{(n+2)s}}\right) \\
&= 1 + \frac{f(p) - f(p)}{p^s} + \sum_{m=2}^{\infty} \frac{f(p^m) - f(p)f(p^{m-1}) + p^{k-1}f(p^{m-2})}{p^{ms}} \\
&= 1.
\end{aligned}$$

Convergence of Dirichlet series

1. (a) The Taylor series for $\ln(1+x)$ is:

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}x}{n}.$$

valid for $|x| < 1$.

Thus **assuming** the sum converges its value should be $\ln(1+1) = \ln(2)$ (we are not guaranteed convergence of the Taylor series since $x = 1$ does not satisfy $|x| < 1$).

We now show that the sum does indeed converge. Consider the Dirichlet series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^s}$. Certainly the values $|1|, |1-1|, |1-1+1|, ...$ are bounded above by 1, hence by Theorem 2.30 the Dirichlet series converges for $\mathrm{Re}(s) > 0$. In particular it converges when $s = 1$, which is the sum in question.

The sum does **not** converge absolutely since:

$$\sum_{n=1}^{\infty} \left|\frac{(-1)^{n+1}}{n}\right| = \sum_{n=0}^{\infty} \frac{1}{n}$$

which is the Harmonic series (known to diverge although we will prove it again later in the course).

(b) Since every $n \geq 1$ is either an odd number, twice an odd number or a multiple of 4 it is clear that every $n \geq 1$ appears **exactly once** as a denominator in the sum defining T. Also the correct sign appears in front of each term hence we have a rearrangement of S.

(c) Notice that:

$$\frac{1}{2k+1} - \frac{1}{2(2k+1)} - \frac{1}{4k} = \frac{1}{2(2k+1)} - \frac{1}{4k} = \frac{1}{2}\left(\frac{1}{2k+1} - \frac{1}{2k}\right).$$

Thus:

$$T = \frac{1}{2}\sum_{k=1}^{\infty}\left(\frac{1}{2k+1} - \frac{1}{2k}\right) = \frac{1}{2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n} = \frac{S}{2} = \frac{\ln(2)}{2}.$$

2. (a) Letting $\mathrm{Re}(s) = \sigma$:

$$\sum_{n=1}^{\infty}\left|\frac{f(n)}{n^s}\right| = \sum_{n=1}^{\infty}\frac{|f(n)|}{n^\sigma} \le \sum_{n=1}^{\infty}\frac{M}{n^\sigma} = M\zeta(\sigma),$$

which by the assumption converges absolutely when $\sigma > 1$.

(b) We know that $D(s, N_\alpha) = \zeta(s-\alpha)$, which converges absolutely for $\mathrm{Re}(s-\alpha) > 1$, so that $\mathrm{Re}(s) > 1 + \mathrm{Re}(\alpha)$.

(c) Letting $\mathrm{Re}(s) = \sigma$:

$$\sum_{n=0}^{\infty}\left|\frac{f(n)}{n^s}\right| = \sum_{n=0}^{\infty}\frac{|f(n)|}{n^\sigma} \le \sum_{n=0}^{\infty}\frac{C}{n^{\sigma-\delta}} = C\zeta(\sigma-\delta)$$

which converges absolutely for $\sigma - \delta > 1$, i.e. $\sigma > 1 + \delta$.
This does indeed imply (b) since $|N_\alpha(n)| = n^{\mathrm{Re}(\alpha)}$ and so we may trivially take $C = 1$ and $\delta = \mathrm{Re}(\alpha)$.

3. If $f(n) \in \mathbb{R}$ and $f(n) > 0$ for all n then $f(n) = |f(n)|$ hence $D(s, f)$ should converge and converge absolutely for the same values of s. So the two abscissae are equal.

MAS430 - Chapter 3 Solutions

Behaviour of $\zeta(s)$.

1. (a) For $\sigma > 1$ and $n \geq 1$ we have:

$$\int_n^{n+1} \frac{dx}{x^\sigma} = \left[\frac{x^{1-\sigma}}{1-\sigma}\right]_n^{n+1} = \frac{(n+1)^{1-\sigma} - n^{1-\sigma}}{1-\sigma} = \frac{n^{1-\sigma} - (n+1)^{1-\sigma}}{\sigma - 1}.$$

(b) For $\sigma > 1$ consider the function $f(x) = \frac{1}{x^\sigma}$. For $n \geq 1$ the maximum value of f in the interval $[n, n+1]$ is $\frac{1}{n^\sigma}$ and the minimum value is $\frac{1}{(n+1)^\sigma}$.
The length of the inverval $[n, n+1]$ is 1.
Thus:

$$\int_n^{n+1} \frac{dx}{x^\sigma} \leq \frac{1}{n^\sigma}$$

and

$$\int_n^{n+1} \frac{dx}{x^\sigma} \geq \frac{1}{(n+1)^\sigma}.$$

The inequality follows.

(c) Summing the inequality for $n = 1, 2, ...$ gives:

$$\zeta(\sigma) - 1 < \frac{1}{\sigma - 1} < \zeta(\sigma).$$

Rearranging gives:

$$\frac{1}{\sigma - 1} < \zeta(\sigma) < \frac{1}{\sigma - 1} + 1.$$

Bernoulli numbers/polynomials and behaviour of ζ.

1. (a) It is easy to check that the substitution transforms S into T.
Note that the determinant of the Jacobian for this transformation is:

$$\begin{aligned} |J| = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} &= \begin{vmatrix} \frac{\cos u}{\cos v} & \frac{\sin u \sin v}{\cos v^2} \\ \frac{\sin u \sin v}{\cos u^2} & \frac{\cos v}{\cos u} \end{vmatrix} \\ &= 1 - \tan u^2 \tan v^2 \\ &= 1 - (xy)^2 \end{aligned}$$

Thus:

$$\int\int_S \frac{dx\,dy}{1-(xy)^2} = \int\int_T \frac{|J|}{1-(xy)^2}\,du\,dv = \int\int_T du\,dv.$$

This integral is equal to the area of the triangle T, which is $\frac{1}{2}\times\frac{\pi}{2}\times\frac{\pi}{2} = \frac{\pi^2}{8}$.

(b) Using the geometric series formula:

$$\frac{1}{1-(xy)^2} = 1 + (xy)^2 + (xy)^4 + ...$$

Performing the double integral term by term (which is valid here) gives:

$$\begin{aligned}\int\int_S \frac{dx\,dy}{1-(xy)^2} &= \int_0^1\int_0^1 (1+(xy)^2+(xy)^4+...)\,dx\,dy \\ &= \int_0^1 (1+\frac{y^2}{3}+\frac{y^4}{5}+...)\,dy \\ &= 1+\frac{1}{3^2}+\frac{1}{5^2}+...\end{aligned}$$

However we have just observed that this double integral equals $\frac{\pi^2}{8}$ and so:

$$1+\frac{1}{3^2}+\frac{1}{5^2}+... = \frac{\pi^2}{8}.$$

(c) Note that by the fact that $\zeta(s)$ is absolutely convergent at $s=2$:

$$\begin{aligned}\zeta(2) &= 1+\frac{1}{2^2}+\frac{1}{3^2}+... \\ &= \left(1+\frac{1}{3^2}+\frac{1}{5^2}+...\right)+\left(\frac{1}{2^2}+\frac{1}{4^2}+\frac{1}{6^2}+...\right) \\ &= \frac{\pi^2}{8}+\frac{\zeta(2)}{4}\end{aligned}$$

Thus $\frac{3\zeta(2)}{4} = \frac{\pi^2}{8}$ and so $\zeta(2) = \frac{\pi^2}{6}$.

2. (a) By definition:

$$\frac{t}{e^t-1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!}.$$

Thus:

$$\begin{aligned}t &= (e^t-1)\sum_{n=0}^{\infty} B_n\frac{t^n}{n!} \\ &= \left(\sum_{m=1}^{\infty}\frac{t^m}{m!}\right)\left(\sum_{n=0}^{\infty}B_n\frac{t^n}{n!}\right) \\ &= \sum_{k=1}^{\infty}\left(\sum_{n=0}^{k-1}\frac{B_n}{n!(k-n)!}\right)t^k \\ &= \sum_{k=1}^{\infty}\left(\sum_{n=0}^{k-1}\binom{k}{n}B_n\right)\frac{t^k}{k!}\end{aligned}$$

Comparing the t^k term on both sides (for $k \geq 2$) gives:

$$\sum_{n=0}^{k-1} \binom{k}{n} B_n = 0.$$

(Comparing the t terms on both sides just gives $B_0 = 1$ which we already knew).

(b) This is not as tedious as you may think since we already know that odd indexed Bernoulli numbers are 0 (except for B_1).
Using the recursion we find:

$$\binom{5}{0} B_0 + \binom{5}{1} B_1 + \binom{5}{2} B_2 + \binom{5}{3} B_3 + \binom{5}{4} B_4 = 0$$

Thus:

$$5B_4 = -B_0 - 5B_1 - 10B_2 - 10B_3 = -1 + \frac{5}{2} - \frac{5}{3} = -\frac{1}{6},$$

giving $B_4 = -\frac{1}{30}$
In a similar vein we find that:

$$7B_6 = -B_0 - 7B_1 - 21B_2 - 35B_3 - 35B_4 - 21B_5 = -1 + \frac{7}{2} - \frac{7}{2} + \frac{7}{6} = \frac{1}{6}$$

giving $B_6 = \frac{1}{42}$.
Continuing to use the recursion you find that $B_8 = -\frac{1}{30}, B_{10} = \frac{5}{66}$ and $B_{12} = -\frac{691}{2730}$.

3. (a) We have by definition that:

$$\frac{te^{xt}}{e^t - 1} = \sum_{k=0}^{\infty} B_k(x) \frac{t^k}{k!}$$

However

$$\begin{aligned}
\left(\frac{t}{e^t - 1}\right) e^{xt} &= \left(\sum_{n=0}^{\infty} B_n \frac{t^n}{n!}\right) \left(\sum_{m=0}^{\infty} \frac{(xt)^m}{m!}\right) \\
&= \sum_{k=0} \left(\sum_{m+n=k} B_n x^m \frac{t^k}{m!n!}\right) \\
&= \sum_{k=0}^{\infty} \left(\sum_{n=0}^{k} B_n x^{k-n} \frac{t^k}{n!(k-n)!}\right)
\end{aligned}$$

Comparing the t^k terms gives:

$$\frac{B_k(x)}{k!} = \sum_{n=0}^{k} \frac{B_n}{n!(k-n)!} x^{k-n}$$

Thus:

$$B_k(x) = \sum_{n=0}^{k} \frac{k!}{n!(k-n)!} B_n x^{k-n} = \sum_{n=0}^{k} \binom{k}{n} B_n x^{k-n}.$$

We use the formula with $B_0 = 1, B_1 = -\frac{1}{2}, B_2 = \frac{1}{6}, B_3 = 0, B_4 = -\frac{1}{30}, B_5 = 0$ to get:

$$\begin{aligned}
B_1(x) &= B_0 x + B_1 \\
&= x - \frac{1}{2} \\
B_2(x) &= B_0 x^2 + 2B_1 x + B_2 \\
&= x^2 - x + \frac{1}{6} \\
B_3(x) &= B_0 x^3 + 3B_1 x^2 + 3B_2 x + B_3 \\
&= x^3 - \frac{3}{2}x^2 + \frac{1}{2}x \\
B_4(x) &= B_0 x^4 + 4B_1 x^3 + 6B_2 x^2 + 4B_3 x + B_4 \\
&= x^4 - 2x^3 + x^2 - \frac{1}{30} \\
B_5(x) &= B_0 x^5 + 5B_1 x^4 + 10B_2 x^3 + 10B_3 x^2 + 5B_4 x + B_5 \\
&= x^5 - \frac{5}{2}x^4 + \frac{5}{3}x^3 - \frac{1}{6}x
\end{aligned}$$

(b) If $x = 1$ we know that $B_k(1) = B_k$. On the other hand the formula gives:

$$B_k = B_k(1) = \sum_{n=0}^{k} \binom{k}{n} B_n.$$

Rearranging gives:

$$\sum_{n=0}^{k-1} \binom{k}{n} B_n = 0$$

which is exactly the identity in Exercise 2.

4. Observe:

$$\frac{te^{(x+1)t}}{e^t - 1} - \frac{te^{xt}}{e^t - 1} = \frac{te^{xt}(e^t - 1)}{e^t - 1} = te^{xt}$$

Now compare both sides as power series.

5. (a) For each integer $1 \leq m \leq n$ we have $B_{k+1}(m+1) - B_{k+1}(m) = (k+1)m^k$.

Summing over each such m gives:

$$\sum_{m=1}^{n} (B_{k+1}(m+1) - B_{k+1}(m)) = (k+1)S_k(n).$$

Evaluating the telescoping sum on the LHS gives:

$$B_{k+1}(n+1) - B_{k+1}(1) = (k+1)S_k(n).$$

Thus:

$$S_k(n) = \frac{B_{k+1}(n+1) - B_{k+1}(1)}{k+1}.$$

Using the formula in Exercise 3 we now have:

$$\begin{aligned} S_k(n) &= \frac{\sum_{r=0}^{k+1}\binom{k+1}{r}B_r\cdot(n+1)^{k+1-r} - B_{k+1}}{k+1} \\ &= \frac{1}{k+1}\sum_{r=0}^{k}\binom{k+1}{r}B_r\cdot(n+1)^{k+1-r}. \end{aligned}$$

(b)

$$\begin{aligned} S_1(n) &= \frac{1}{2}\left(\binom{2}{0}B_0\cdot(n+1)^2+\binom{2}{1}B_1\cdot(n+1)\right) \\ &= \frac{1}{2}((n^2+2n+1)-(n+1)) \\ &= \frac{n(n+1)}{2} \\ S_2(n) &= \frac{1}{3}\left(\binom{3}{0}B_0\cdot(n+1)^3+\binom{3}{1}B_1\cdot(n+1)^2+\binom{3}{2}B_2\cdot(n+1)\right) \\ &= \frac{1}{3}\left((n^3+3n^2+3n+1)-\frac{3}{2}(n^2+2n+1)+\frac{1}{2}(n+1)\right) \\ &= \frac{1}{6}(2n^3+3n^2+n) \\ &= \frac{n(n+1)(2n+1)}{6} \end{aligned}$$

Similarly:

$$\begin{aligned} S_3(n) &= \frac{n^2(n+1)^2}{4} \\ &= \left(\frac{n(n+1)}{2}\right)^2 \\ S_4(n) &= \frac{6n^5+15n^4+10n^3-n}{30} \\ &= \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30} \end{aligned}$$

When $n = 4$

$$\begin{aligned} S_1(4) &= 1+2+3+4 = 10 \\ S_2(4) &= 1+4+9+16 = 30 \\ S_3(4) &= 1+8+27+64 = 100 \\ S_4(4) &= 1+16+81+256 = 354 \end{aligned}$$

It is simple to check that plugging $n = 4$ into the above formulae gives exactly these values.

6. We have that:

$$\zeta(2k) = (-1)^{k+1}\frac{B_{2k}(2\pi)^{2k}}{2(2k)!}.$$

Thus using the values found in Exercise 2:

$$\zeta(4) = -\frac{16B_4\pi^4}{48} = \frac{\pi^4}{90}$$
$$\zeta(6) = \frac{64B_6\pi^6}{1440} = \frac{\pi^6}{945}$$
$$\zeta(8) = -\frac{256B_8\pi^8}{80640} = \frac{\pi^8}{9450}$$
$$\zeta(10) = \frac{1024B_{10}\pi^{10}}{7257600} = \frac{\pi^{10}}{93555}$$
$$\zeta(12) = -\frac{4096B_{12}\pi^{12}}{958003200} = \frac{691\pi^{12}}{638512875}$$

7. (a) Taking logs gives:

$$\ln(\sin z) - \ln(z) = \sum_{n=1}^{\infty} \ln\left(1 - \frac{z^2}{(\pi n)^2}\right).$$

Differentiating this with respect to z reveals:

$$\frac{\cos z}{\sin z} - \frac{1}{z} = -2\sum_{n=1}^{\infty} \frac{\frac{z}{(\pi n)^2}}{1 - \frac{z^2}{(\pi n)^2}}.$$

Thus:

$$\begin{aligned}\frac{z\cos z}{\sin z} &= 1 - 2\sum_{n=1}^{\infty} \frac{\frac{z^2}{(\pi n)^2}}{1 - \frac{z^2}{(\pi n)^2}} \\ &= 1 - 2\sum_{n=1}^{\infty} \left(\frac{z^2}{(\pi n)^2} + \frac{z^4}{(\pi n)^4} + ...\right) \\ &= 1 - 2\sum_{k=1}^{\infty} \frac{\zeta(2k)}{\pi^{2k}} z^{2k}\end{aligned}$$

(b) Recall:

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}$$
$$\cos z = \frac{e^{iz} + e^{-iz}}{2}$$

Thus:

$$\begin{aligned}\frac{z\cos z}{\sin z} &= \frac{iz(e^{iz} + e^{-iz})}{(e^{iz} - e^{-iz})} \\ &= \frac{iz(e^{2iz} + 1)}{(e^{2iz} - 1)} \\ &= \frac{iz(e^{2iz} - 1 + 2)}{(e^{2iz} - 1)} \\ &= iz + \frac{2iz}{e^{2iz} - 1} \\ &= iz + \sum_{m=0}^{\infty} B_m \frac{(2iz)^m}{m!}\end{aligned}$$

Now the $m = 0$ term in the sum is 1 and the $m = 1$ term in the sum is $-iz$. Also odd index Bernoulli numbers are 0, so:

$$\frac{z\cos z}{\sin z} = 1 + \sum_{k=1}^{\infty} B_{2k}\frac{(2iz)^{2k}}{(2k)!}$$
$$= 1 + \sum_{k=1}^{\infty}(-1)^k \frac{B_{2k}2^{2k}}{(2k)!}z^{2k}.$$

Now comparing this with the expansion found in (a) we see that for $k \geq 1$:

$$-\frac{2\zeta(2k)}{\pi^{2k}} = (-1)^k \frac{B_{2k}2^{2k}}{(2k)!}.$$

Rearranging gives:

$$\zeta(2k) = (-1)^{k+1}\frac{B_{2k}(2\pi)^{2k}}{2(2k)!}.$$

8. (a) Recall that the Fourier expansion of $P_{2k+1}(x)$ is:

$$P_{2k+1}(x) = (-1)^{k+1}(2k+1)!\sum_{n=1}^{\infty}\frac{2}{(2n\pi)^{2k+1}}\sin(2n\pi x).$$

Since each $P_{2k+1}(x)$ is continuous at $x = \frac{1}{4}$ we may substitute this in to get:

$$P_{2k+1}\left(\frac{1}{4}\right) = (-1)^{k+1}(2k+1)!\sum_{n=1}^{\infty}\frac{2}{(2n\pi)^{2k+1}}\sin\left(\frac{n\pi}{2}\right).$$

Now if n is even then $\sin(\frac{n\pi}{2}) = 0$. If $n \equiv 1 \bmod 4$ then $\sin(\frac{n\pi}{2}) = 1$ and if $n \equiv 3 \bmod 4$ then $\sin(\frac{n\pi}{2}) = -1$.
Thus:

$$P_{2k+1}\left(\frac{1}{4}\right) = \frac{(-1)^{k+1}(2k+1)!}{2^{2k}\pi^{2k+1}}\left(1 - \frac{1}{3^{2k+1}} + \frac{1}{5^{2k+1}} - ...\right)$$

from which it is apparent that:

$$\left(1 - \frac{1}{3^{2k+1}} + \frac{1}{5^{2k+1}} - ...\right) = \frac{(-1)^{k+1}2^{2k}\pi^{2k+1}P_{2k+1}\left(\frac{1}{4}\right)}{(2k+1)!}.$$

(b) Since $P_{2k+1}\left(\frac{1}{4}\right) = B_{2k+1}\left(\frac{1}{4}\right)$ and:

$$B_1(x) = x - \frac{1}{2}$$

$$B_3(x) = x^3 - \frac{3}{2}x^2 + \frac{1}{2}x$$

$$B_5(x) = x^5 - \frac{5}{2}x^4 + \frac{5}{3}x^3 - \frac{1}{6}x$$

we see that:

$$S_0 = 1 - \frac{1}{3} + \frac{1}{5} - ... = (-1)\pi B_1\left(\frac{1}{4}\right) = \frac{\pi}{4}$$

$$S_1 = 1 - \frac{1}{3^3} + \frac{1}{5^3} - ... = \frac{4\pi^3 B_3\left(\frac{1}{4}\right)}{3!} = \frac{2\pi^3\left(\frac{3}{64}\right)}{3} = \frac{\pi^3}{32}.$$

$$S_2 = 1 - \frac{1}{3^5} + \frac{1}{5^5} - ... = \frac{(-1)2^4\pi^5 B_5\left(\frac{1}{4}\right)}{5!} = \frac{2^4\pi^5\left(\frac{25}{1024}\right)}{120} = \frac{5\pi^5}{1536}.$$

MAS430 - Chapter 4 Solutions

Characters

1. (a) $(\mathbb{Z}/3\mathbb{Z})^\times = \{\overline{1}, \overline{2}\} = \langle \overline{2} \rangle$,
$(\mathbb{Z}/7\mathbb{Z})^\times = \{\overline{1}, \overline{2}, \overline{3}, \overline{4}, \overline{5}, \overline{6}\} = \langle \overline{3} \rangle$,
$(\mathbb{Z}/10\mathbb{Z})^\times = \{\overline{1}, \overline{3}, \overline{7}, \overline{9}\} = \langle \overline{3} \rangle$,
$(\mathbb{Z}/24\mathbb{Z})^\times = \{\overline{1}, \overline{5}, \overline{7}, \overline{11}, \overline{13}, \overline{17}, \overline{19}, \overline{23}\} = \langle \overline{5}, \overline{7}, \overline{13} \rangle$.

   (b) Let χ be a character of $(\mathbb{Z}/3\mathbb{Z})^\times = \langle \overline{2} \rangle$. Then since $\overline{2}$ has order 2 we know that $\chi(\overline{2}) \in \{\pm 1\}$. Thus we get 2 characters:

	$\overline{1}$	$\overline{2}$
χ_0	1	1
χ_1	1	-1

Let χ be a character of $(\mathbb{Z}/7\mathbb{Z})^\times = \langle \overline{3} \rangle$. Then since $\overline{3}$ has order 6 we know that $\chi(\overline{3}) \in \{1, \zeta, \zeta^2, \zeta^3, \zeta^4, \zeta^5\}$, where $\zeta = e^{\frac{2\pi i}{6}} = e^{\frac{\pi i}{3}} = \frac{1+i\sqrt{3}}{2}$ is a primitive 6th root of unity. Thus there are 6 characters (notice that $\zeta^3 = -1$):

	$\overline{1}$	$\overline{2}$	$\overline{3}$	$\overline{4}$	$\overline{5}$	$\overline{6}$
χ_0	1	1	1	1	1	1
χ_1	1	ζ^2	ζ	ζ^4	ζ^5	ζ^3
χ_2	1	ζ^4	ζ^2	ζ^2	ζ^4	1
χ_3	1	1	-1	1	-1	-1
χ_4	1	ζ^2	ζ^4	ζ^4	ζ^2	1
χ_5	1	ζ^4	ζ^5	ζ^2	ζ	ζ^3

Let χ be a character of $(\mathbb{Z}/10\mathbb{Z})^\times = \langle \overline{3} \rangle$. Then since $\overline{3}$ has order 4 we know that $\chi(\overline{3}) \in \{\pm 1, \pm i\}$. Thus there are 4 characters:

	$\overline{1}$	$\overline{3}$	$\overline{7}$	$\overline{9}$
χ_0	1	1	1	1
χ_1	1	i	$-i$	-1
χ_2	1	-1	-1	1
χ_3	1	$-i$	i	-1

Let χ be a character of $(\mathbb{Z}/24\mathbb{Z})^\times = \langle \overline{5}, \overline{7}, \overline{13} \rangle$. Then since $\overline{5}$ has order 2 we know that $\chi(\overline{5}) \in \{\pm 1\}$. Similarly for $\chi(\overline{7})$ and $\chi(\overline{13})$. Thus there are 8 characters:

	$\overline{1}$	$\overline{5}$	$\overline{7}$	$\overline{11}$	$\overline{13}$	$\overline{17}$	$\overline{19}$	$\overline{23}$
χ_0	1	1	1	1	1	1	1	1
χ_1	1	-1	1	-1	1	-1	1	-1
χ_2	1	1	-1	-1	1	1	-1	-1
χ_3	1	1	1	1	-1	-1	-1	-1
χ_4	1	-1	-1	1	1	-1	-1	1
χ_5	1	-1	1	-1	-1	1	-1	1
χ_6	1	1	-1	-1	-1	-1	1	1
χ_7	1	-1	-1	1	-1	1	1	-1

(c) One easily checks the orthogonality relations by summing the rows and columns of these tables. It is observed that the entries in the first row/column always add up to $\phi(m)$ and in all other row/columns the entries add up to 0.

2. (a) Clearly:

$$G(0,\chi) = \sum_{k=1}^{m} \chi(k) = \sum_{\overline{k}\in(\mathbb{Z}/m\mathbb{Z})^{\times}} \chi(\overline{k}).$$

By the orthogonality relations this is $\phi(m)$ if χ is trivial and 0 otherwise.

(b) Note that:

$$G(a,\chi) = \sum_{\overline{k}\in(\mathbb{Z}/m\mathbb{Z})^{\times}} \chi(\overline{k})\zeta_m^{ak}.$$

Now as $\overline{k}$ runs through all elements of $(\mathbb{Z}/m\mathbb{Z})^{\times}$ so does $\overline{ak}$ (since a is coprime to m).
Hence we may change variables $\overline{b} = \overline{ak}$.

$$\begin{aligned} G(a,\chi) &= \sum_{\overline{b}\in(\mathbb{Z}/m\mathbb{Z})^{\times}} \chi(\overline{a}^{-1}\overline{b})\zeta_m^{b} \\ &= \chi(\overline{a})^{-1} \sum_{\overline{b}\in(\mathbb{Z}/m\mathbb{Z})^{\times}} \chi(\overline{b})\zeta_m^{b} \\ &= \overline{\chi(a)}G(1,\chi) \end{aligned}$$

(c) By definition $\zeta_p^p - 1 = 0$ so that by factorising:

$$(\zeta_p - 1)(\zeta_p^{p-1} + \zeta_p^{p-2} + ... + \zeta_p + 1) = 0.$$

But $\zeta_p \neq 1$ hence $\zeta_p^{p-1} + \zeta_p^{p-2} + ... + \zeta_p + 1 = 0$. We will use this identity throughout the calculations.

$$\begin{aligned} G(1,\chi_3)^2 &= (\zeta_3 - \zeta_3^2)^2 \\ &= \zeta_3^2 - 2\zeta_3^3 + \zeta_3^4 \\ &= (\zeta_3 + \zeta_3^2) - 2 \\ &= -1 - 2 \\ &= -3 \end{aligned}$$

$$\begin{aligned} G(1,\chi_5)^2 &= (\zeta_5 - \zeta_5^2 - \zeta_5^3 + \zeta_5^4)^2 \\ &= \zeta_5^2 - 2\zeta_5^3 - \zeta_5^4 + 4\zeta_5^5 - \zeta_5^6 - 2\zeta_5^7 + \zeta_5^8 \\ &= -(\zeta_5 + \zeta_5^2 + \zeta_5^3 + \zeta_5^4) + 4 \\ &= 1 + 4 \\ &= 5 \end{aligned}$$

$$\begin{aligned} G(1,\chi_7)^2 &= (\zeta_7 + \zeta_7^2 - \zeta_7^3 + \zeta_7^4 - \zeta_7^5 - \zeta_7^6)^2 \\ &= \zeta_7^2 + 2\zeta_7^3 - \zeta_7^4 + \zeta_5^6 - 6\zeta_7^7 + \zeta_7^8 - \zeta_7^{10} + 2\zeta_7^{11} + \zeta_7^{12} \\ &= (\zeta_7 + \zeta_7^2 + \zeta_7^3 + \zeta_7^4 + \zeta_7^5 + \zeta_7^6) - 6 \\ &= -1 - 6 \\ &= -7 \end{aligned}$$

(d) By a valid change of variables:

$$\begin{aligned} G(1,\chi_p)^2 &= \left(\sum_{s=1}^{p-1}\left(\frac{s}{p}\right)\zeta_p^s\right)\left(\sum_{t=1}^{p-1}\left(\frac{t}{p}\right)\zeta_p^t\right) \\ &= \sum_{s=1}^{p-1}\sum_{t=1}^{p-1}\left(\frac{st}{p}\right)\zeta_p^{s+t} \\ &= \sum_{s=1}^{p-1}\sum_{t'=1}^{p-1}\left(\frac{s^2t'}{p}\right)\zeta_p^{s+st'} \\ &= \sum_{s=1}^{p-1}\sum_{t'=1}^{p-1}\left(\frac{t'}{p}\right)\zeta_p^{s(1+t')} \\ &= \sum_{t'=1}^{p-1}\left(\sum_{s=1}^{p-1}\zeta_p^{s(1+t')}\right)\left(\frac{t'}{p}\right) \end{aligned}$$

Recall the identity in (b):

$$\zeta_p^{p-1} + \zeta_p^{p-2} + ... + \zeta_p + 1 = 0.$$

Now for $t' = 1, 2, ..., p-2$ the inner sum above is -1 by the identity (since for these t' we have $1 + t' \not\equiv 0 \bmod p$ and so as s runs from $1, 2, ..., p-1 \bmod p$ so does $s(1+t')$).

For $t' = p-1$ the inner sum is clearly $p-1$ since $\zeta_p^{s(1+t')} = \zeta_p^{sp} = 1$.

So:

$$\begin{aligned}
G(1,\chi_p)^2 &= -\sum_{t'=1}^{p-2}\left(\frac{t'}{p}\right)+\left(\frac{p-1}{p}\right)(p-1)\\
&= -\sum_{t'=1}^{p-1}\left(\frac{t'}{p}\right)+\left(\frac{p-1}{p}\right)p\\
&= \left(\frac{p-1}{p}\right)p\\
&= \left(\frac{-1}{p}\right)p\\
&= (-1)^{\frac{p-1}{2}}p
\end{aligned}$$

by the orthogonality relations.

Dirichlet L series and primes in arithmetic progressions

1. (a) Using the notation of the solution to the earlier exercise:

$$\mathrm{Re}(L(1,\chi_1)) = \left(1-\frac{1}{9}\right)+\left(\frac{1}{11}-\frac{1}{19}\right)+\left(\frac{1}{21}-\frac{1}{29}\right)+\left(\frac{1}{31}-\frac{1}{39}\right)+...$$

Clearly each bracket is positive hence $\mathrm{Re}(L(1,\chi_1)) > 0$. Thus $L(1,\chi_1) \neq 0$.
Since $\mathrm{Re}(L(1,\chi_3)) = \mathrm{Re}(L(1,\chi_1))$ we also know that $L(1,\chi_3) \neq 0$.
Finally, since χ_2 is real valued we must consider the full sum:

$$L(1,\chi_2) = 1-\frac{1}{3}-\frac{1}{7}+\left(\frac{1}{9}+\frac{1}{11}-\frac{1}{13}-\frac{1}{17}\right)+\left(\frac{1}{19}+\frac{1}{21}-\frac{1}{23}-\frac{1}{27}\right)+...$$

Clearly each bracket is positive hence $L(1,\chi_2) > 0$, so that $L(1,\chi_2) \neq 0$.

(b) If $p = 2, 5$ then the sum is clearly 0 since $\chi(p) = 0$. Otherwise p is coprime to 10.
Note that:

$$\sum_{i=0}^{3}\chi_i(3)^{-1}\chi_i(p) = \chi_0(p) - i\chi_1(p) - \chi_2(p) + i\chi_3(p).$$

If $p \equiv 1 \bmod 10$ then:

$$\chi_0(p) - i\chi_1(p) - \chi_2(p) - +i\chi_3(p) = 1 - i - 1 + i = 0.$$

If $p \equiv 3 \bmod 10$ then:

$$\chi_0(p) - i\chi_1(p) - \chi_2(p) + i\chi_3(p) = 1 - i^2 + (-1)^2 - i^2 = 4.$$

If $p \equiv 7 \bmod 10$ then:

$$\chi_0(p) - i\chi_1(p) - \chi_2(p) + i\chi_3(p) = 1 + (-i)^2 + (-1)^2 + i^2 = 0.$$

If $p \equiv 9 \bmod 10$ then:

$$\chi_0(p) - i\chi_1(p) - \chi_2(p) + i\chi_3(p) = 1 + i - 1 - i = 0.$$

(c) We know that $L(s, \chi_0)$ is convergent and analytic on $\mathrm{Re}(s) > 0$ except for a pole at $s = 1$.
For $i = 1, 2, 3$ we know that $L(s, \chi_i)$ is convergent and analytic on $\mathrm{Re}(s) > 0$ and that $L(1, \chi_i) \neq 0$ (by part (a)).
Then for $i = 0, 1, 2, 3$ we may take logs to get:

$$\ln(L(s, \chi_i)) = \sum_{p \neq 2,5} \frac{\chi_i(p)}{p^s} + A(s, \chi_i),$$

where:

$$A(s, \chi_i) = \sum_{p \neq 2,5} \sum_{n=2}^{\infty} \frac{\chi_i(p)}{np^{ns}}.$$

For each i we know that $A(s, \chi_i)$ converges as $s \to 1$.
Now consider the linear combination:

$$\sum_{i=0}^{3} \chi_i(3)^{-1} \ln(L(s, \chi_i))$$

This diverges as $s \to 1$.
However:

$$\sum_{i=1}^{3} \chi_i(3)^{-1} \left(\sum_{p \neq 2,5} \frac{\chi_i(p)}{p^s} + A(s, \chi_i) \right) = 4 \sum_{p \equiv 3 \bmod 10} \frac{1}{p^s} + \sum_{i=1}^{3} \chi_i(3)^{-1} A(s, \chi_i).$$

Letting $s \to 1$ we see that:

$$4 \sum_{p \equiv 3 \bmod 10} \frac{1}{p}$$

diverges, hence there are infinitely many primes congruent to 3 mod 10.

2. (a)

$$\begin{aligned}
f_{12}'(x) &= \sum_{k=0}^{\infty} \left(x^{12k} - x^{12k+4} - x^{12k+6} + x^{12k+10} \right) \\
&= (1 - x^4 - x^6 + x^{10}) \sum_{k=0}^{\infty} x^{12k} \\
&= \frac{1 - x^4 - x^6 + x^{10}}{1 - x^{12}} \\
&= \frac{(x^2 - 1)(x^8 + x^6 - x^2 - 1)}{(-x^4 + x^2 - 1)(x^8 + x^6 - x^2 - 1)} \\
&= \frac{1 - x^2}{x^4 - x^2 + 1}
\end{aligned}$$

(b) Note the factorisation:

$$x^4 - x^2 + 1 = (x^2 + \sqrt{3}x + 1)(x^2 - \sqrt{3}x + 1).$$

Thus we have a partial fraction expansion of the form:

$$f_{12}'(x) = \frac{1 - x^2}{x^4 - x^2 + 1} = \frac{Ax + B}{x^2 + \sqrt{3}x + 1} + \frac{Cx + D}{x^2 - \sqrt{3}x + 1}.$$

After doing the usual calculations one finds:

$$f_{12}'(x) = \frac{1}{2\sqrt{3}} \left(\frac{2x + \sqrt{3}}{x^2 + \sqrt{3}x + 1} - \frac{2x - \sqrt{3}}{x^2 - \sqrt{3}x + 1} \right).$$

Thus integrating gives:

$$\begin{aligned} f_{12}(x) &= \frac{1}{2\sqrt{3}} (\ln(x^2 + \sqrt{3}x + 1) - \ln(x^2 - \sqrt{3}x + 1)) + C \\ &= \frac{1}{2\sqrt{3}} \ln \left(\frac{x^2 + \sqrt{3}x + 1}{x^2 - \sqrt{3}x + 1} \right) + C. \end{aligned}$$

(c) Since and $L(1, \chi_{12}) = f_{12}(1) = f_{12}(1) - f_{12}(0)$ we have:

$$\begin{aligned} L(1, \chi_{12}) &= \frac{1}{2\sqrt{3}} \left[\ln \left(\frac{2 + \sqrt{3}}{2 - \sqrt{3}} \right) - \ln(1) \right] + C - C \\ &= \frac{1}{2\sqrt{3}} \ln((2 + \sqrt{3})^2) \\ &= \frac{1}{\sqrt{3}} \ln(2 + \sqrt{3}) \end{aligned}$$

(d) It is easy to see that $(2 + \sqrt{3})(2 - \sqrt{3}) = 2^2 - 3 \cdot 1^2 = 1$ so that $(2 + \sqrt{3})$ is a unit of $\mathbb{Z}[\sqrt{3}]$ and that $(2, 1)$ is a solution to Pell's equation $x^2 - 3y^2 = 1$.

3. We will define:

$$f_8(x) = \sum_{k=1}^{\infty} \left(\frac{x^{8k+1}}{8k + 1} - \frac{x^{8k+3}}{8k + 3} - \frac{x^{8k+5}}{8k + 5} + \frac{x^{8k+7}}{8k + 7} \right).$$

Then:

$$\begin{aligned} f_8'(x) &= \sum_{k=0}^{\infty} (x^{8k} - x^{8k+2} - x^{8k+4} + x^{8k+6}) \\ &= (1 - x^2 - x^4 + x^6) \sum_{k=0}^{\infty} x^{8k} \\ &= \frac{1 - x^2 - x^4 + x^6}{1 - x^8} \\ &= \frac{(x^2 - 1)(x^4 - 1)}{(1 + x^4)(1 - x^4)} \\ &= \frac{1 - x^2}{x^4 + 1} \end{aligned}$$

Note the factorisation:

$$x^4 + 1 = (x^2 + \sqrt{2}x + 1)(x^2 - \sqrt{2}x + 1)$$

Thus we have a partial fraction expansion of the form:

$$f_8'(x) = \frac{1 - x^2}{x^4 + 1} = \frac{Ax + B}{x^2 + \sqrt{2}x + 1} + \frac{Cx + D}{x^2 - \sqrt{2}x + 1}.$$

After calculation one finds:

$$f_8'(x) = \frac{1}{2\sqrt{2}} \left(\frac{2x + \sqrt{2}}{x^2 + \sqrt{2}x + 1} - \frac{2x - \sqrt{2}}{x^2 - \sqrt{2}x + 1} \right).$$

Thus integrating gives:

$$\begin{aligned} f_8(x) &= \frac{1}{2\sqrt{2}} \Big(\ln(x^2 + \sqrt{2}x + 1) - \ln(x^2 - \sqrt{2}x + 1) \Big) + C \\ &= \frac{1}{2\sqrt{2}} \ln \left(\frac{x^2 + \sqrt{2}x + 1}{x^2 - \sqrt{2}x + 1} \right) + C \end{aligned}$$

Since $L(1, \chi_8) = f_8(1) = f_8(1) - f_8(0)$ we have:

$$\begin{aligned} L(1, \chi_8) &= \frac{1}{2\sqrt{2}} \left[\ln \left(\frac{2 + \sqrt{2}}{2 - \sqrt{2}} \right) - \ln(1) \right] + C - C \\ &= \frac{1}{2\sqrt{2}} \ln \left(\frac{(2 + \sqrt{2})^2}{2} \right) \\ &= \frac{1}{2\sqrt{2}} \ln(3 + 2\sqrt{2}) \end{aligned}$$

It is easy to see that $(3+2\sqrt{2})(3-2\sqrt{2}) = 3^2 - 2 \cdot 2^2 = 1$ so that $(3+2\sqrt{2})$ is a unit of $\mathbb{Z}[\sqrt{2}]$ and that $(3, 2)$ is a solution to Pell's equation $x^2 - 2y^2 = 1$.