The magic of modular forms

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Outline of talk

1. Lattices and the kissing number
2. Modular Forms
3. Return to the kissing number

Kissing number problem

Given a unit sphere in $\mathbb{R}^n$, what is the maximum number of non-overlapping unit spheres I can place around it, each touching the original sphere?

We only know exact answers for dimensions $1, 2, 3, 4, 8$ and 24! In this talk we will see how modular forms become useful in providing the answer in 24 dimensions.

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A lattice is a discrete subset $\Lambda \subseteq \mathbb{R}^n$ of the form:

$$\Lambda = \mathbb{Z}v_1 \oplus \mathbb{Z}v_2 \oplus \dots \oplus \mathbb{Z}v_n,$$

for some basis $v_1, v_2, \dots, v_n$ of $\mathbb{R}^n$.

Lattices provide regular sphere packings (take the coordinates of each lattice vector as the center of a sphere).

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A modular form of weight $k \in \mathbb{N}$ is a holomorphic function $f : \mathcal{H} \to \mathbb{C}$ such that:

- $f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z)$ for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$,
- f is "holomorphic at infinity".

In particular $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ and so $f(z+1) = f(z)$ for all z.

Thus modular forms have Fourier expansions in $q = e^{2\pi i z}$:

$$f(z) = a_0 + a_1 q + a_2 q^2 + \ldots$$

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Why do number theorists care about modular forms?

- Fourier coefficients often contain number theoretic data (power divisor sums, values of Riemann Zeta function etc).
- They have many connections with other objects. For example their links with elliptic curves allowed Fermat's Last Theorem to be proved.
- Simple to compute with: the set of modular forms of weight k forms a finite dimensional $\mathbb{C}$-vector space, M_k.

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Examples include:

- Eisenstein series. For even $k \geq 4$:

$$E_k = \frac{1}{2\zeta(k)} \sum_{(m,n)\in\mathbb{Z}^2\setminus(0,0)} \frac{1}{(mz+n)^k}$$
$$= 1 - \frac{2k}{B_k} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n \in M_k$$

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- The Discriminant function:

$$\Delta(z) = q \prod_{n=1}^{\infty} (1-q^n)^{24}$$
$$= \sum_{n=1}^{\infty} \tau(n) q^n \in M_{12},$$

where $\tau(n)$ is known as Ramanujan's tau function.

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A lattice Λ is integral if $\langle u, v\rangle \in \mathbb{Z}$ for all $u, v \in \Lambda$. Further Λ is even if $||v||^2 \in 2\mathbb{Z}$.

Given an integral lattice we may associate to it a book-keeping device for the norms called a theta series:

$$\theta_\Lambda(z) = \sum_{n=0}^{\infty} r_\Lambda(n) q^{\frac{n}{2}},$$

where $r_\Lambda(n)$ is the number of $v \in \Lambda$ such that $||v||^2 = n$.

It is clear that the kissing number K_Λ is then the first non-zero $r_\Lambda(n)$ for $n \geq 1$.

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For example:

$$\theta_{\mathbb{Z}}(z) = 1 + 2\sum_{n=1}^{\infty} q^{\frac{n^2}{2}} = 1 + 2q^{\frac{1}{2}} + 2q^2 + 2q^{\frac{9}{2}} + ...$$

$$\theta_{\mathbb{Z}(1,0)\oplus\mathbb{Z}(0,1)}(z) = 1 + \sum_{n=1}^{\infty} r_2(n) q^{\frac{n}{2}} = 1 + 4q^{\frac{1}{2}} + 4q + ...$$

(where $r_2(n)$ is the number of ways of expressing n as a sum of two squares).

If $\Lambda \subseteq \mathbb{R}^k$ is a "unimodular", even, integral lattice (from now on referred to as "nice") then $\theta_\Lambda(z) \in M_{\frac{k}{2}}$

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It turns out that in 24 dimensions there exists a “nice” lattice Λ_{Leech} such that $r_{\Lambda_{Leech}}(0) = 1$ and $r_{\Lambda_{Leech}}(2) = 0$.

By the power of modular forms this is actually enough information to work out $r_{\Lambda_{Leech}}(n)$ for all n.

Why? Well we happen to know that:

$$\theta_{\Lambda_{Leech}}(z) \in M_{12} = \mathbb{C}E_{12} \oplus \mathbb{C}\Delta.$$

Thus $\theta_{\Lambda_{Leech}}(z) = \alpha E_{12} + \beta\Delta$ for some $\alpha, \beta \in \mathbb{C}$. Comparing the first two coefficients of the Fourier series of both sides gives $\alpha = 1$ and $\beta = -\frac{65520}{691}$.

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Thus for all $n \geq 1$:

$$r_{\Lambda_{Leech}}(2n) = \frac{65520(\sigma_{11}(n) - \tau(n))}{691}.$$

In particular $r_{\Lambda_{Leech}}(4) = 196560 \neq 0$ so we must have $K_{\Lambda_{Leech}} = 196560$.

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Thanks for listening!