The inversion formula for Hilbert-Siegel theta series.

Dan Fretwell

Let K be a totally real number field of degree d with ring of integers $\mathcal{O}_K$ and integral basis $\gamma_1, ..., \gamma_d$. We fix an ordering of real embeddings $\psi_1, ..., \psi_d : K \hookrightarrow \mathbb{R}$. The conjugates of $\alpha \in K$ will be written as $\alpha^{(t)} := \psi_t(\alpha)$ for ease of notation. The maps ψ_t induce linear embeddings $M_{s,t}(K) \hookrightarrow M_{s,t}(\mathbb{R})$ for any s, t by applying ψ_t component-wise. Again for ease of notation we write $U^{(t)} = \psi_t(U)$ for $U \in M_{s,t}(K)$ and refer to these as the conjugates of U.

For a fixed choice of $n \geq 1$ we set $V = M_{m,n}(K)$. Let $Q \in M_m(K)$ be a symmetric, totally positive definite matrix (corresponding to a quadratic form on K^m). We may scale Q so that it is integral and has even diagonals. Given $U, W \in V$ we write $Q[U, W] = U^T Q W \in M_n(K)$ and $Q[U] = U^T Q U \in M_n(K)$. Note that $Q[U]$ is a symmetric matrix.

Considering V as both a K-vector space and a $\mathbb{Q}$-vector space we get quadratic forms $Q_{V,K}, Q_{V,\mathbb{Q}}$ respectively, given by

$$\begin{aligned} Q_{V,K}(U) &= \sigma(Q[U]) \\ Q_{V,\mathbb{Q}}(U) &= \sigma(\mathrm{tr}(Q[U])) = \sigma\left(\sum_{t=1}^{d} Q[U]^{(t)}\right), \end{aligned}$$

where σ is the matrix trace. This gives bilinear forms:

$$\begin{aligned} B_{V,K}(U, W) &= \sigma(Q[U, W]) \\ B_{V,\mathbb{Q}}(U, W) &= \sigma(\mathrm{tr}(Q[U, W])) = \sigma\left(\sum_{t=1}^{d} Q[U, W]^{(t)}\right). \end{aligned}$$

Let $\tau = (\tau^{(1)}, ..., \tau^{(d)}) \in \mathcal{H}_n^d$ (Siegel upper half space of degree n). Abusing notation slightly we will write:

$$Q_{V,\mathbb{Q}}(U\tau) = \sigma(\mathrm{tr}(Q[U]\tau)) = \sigma\left(\sum_{t=1}^{d} Q[U]^{(t)}\tau^{(t)}\right).$$

We will show that there is a matrix $Z_1(\tau) \in \mathcal{H}_{mnd}$ and an isomorphism $\phi' : V \to \mathbb{Q}^{mnd}$ satisfying:

$$Q_{V,\mathbb{Q}}(U\tau) = \phi'(U)^T Z_1(\tau) \phi'(U),$$

i.e. that $Q_{V,\mathbb{Q}}$ is isometric to the quadratic form on $\mathbb{Q}^{mnd}$ with matrix $Z_1(I)$. For $n = 1$ this is precisely the idea behind Eichler's trick for establishing the inversion formula for Hilbert theta series.

Before stating the result we must make a few definitions. First we construct the matrix $G = (\gamma_j^{(i)}) \in M_d(\mathbb{R})$ of basis conjugates and take the Kronecker product $G' = I_n \otimes G \otimes I_m \in M_{mnd}(\mathbb{R})$.

Consider the linear map $\phi : K^m \to \mathbb{Q}^{md}$ that sends column vector

$$\mathbf{u} = (u_1, ..., u_m)^T = \sum_{i=1}^{m} u_i \mathbf{e}_i = \sum_{i,j} u_{i,j} \mathbf{e}_i \gamma_j$$

to

$$(u_{1,1}, u_{2,1}, ..., u_{m,1}, u_{1,2}, ..., u_{m,2}, ..., u_{m,d}) \in \mathbb{Q}^{md}.$$

This map extends to a linear map $\phi : V \to \mathbb{Q}^{mnd}$ via $\phi(U) = (\phi(\mathbf{u}_1), ..., \phi(\mathbf{u}_n))$, where $\mathbf{u}_1, ..., \mathbf{u}_n$ are the columns of U. Let $\phi'(U) = \phi(U)^T$.

We see that there is a strong link between the conjugates of $U \in V$ and the vector $G'\phi'(U) \in \mathbb{Q}^{mnd}$.

Lemma 0.1. *For any $U \in V$ with columns $\mathbf{u}_1, ..., \mathbf{u}_n$ we have*

$$G'\phi'(U) = (\mathbf{v}_1^{(1)}, \mathbf{v}_1^{(2)}, ..., \mathbf{v}_1^{(d)}, \mathbf{v}_2^{(1)}, ..., \mathbf{v}_2^{(d)}, ..., \mathbf{v}_n^{(d)})^T,$$

where $\mathbf{v}_t = \mathbf{u}_t^T$.

Proof. Note that:

$$G'\phi'(U) = \begin{pmatrix} (G \otimes I_m)\phi(\mathbf{u}_1)^T \\ (G \otimes I_m)\phi(\mathbf{u}_2)^T \\ \cdots \\ (G \otimes I_m)\phi(\mathbf{u}_n)^T \end{pmatrix}.$$

It suffices to show that $(G \otimes I_m)\phi(\mathbf{u}_t)^T = (\mathbf{v}_t^{(1)}, ..., \mathbf{v}_t^{(d)})^T$ for each $1 \leq t \leq n$. Letting $\phi(\mathbf{u}_t) = (u_{1,1,t}, u_{2,1,t}, ..., u_{m,1,t}, u_{1,2,t}, ..., u_{m,2,t}, ..., u_{m,d,t})$ the claim follows since:

$$\begin{aligned}(G \otimes I_m)\phi(\mathbf{u}_t)^T = \begin{pmatrix} \sum_{j=1}^{d} \gamma_j^{(1)} \begin{pmatrix} u_{1,j,t} \\ u_{2,j,t} \\ \cdots \\ u_{m,j,t} \end{pmatrix} \\ \sum_{j=1}^{d} \gamma_j^{(2)} \begin{pmatrix} u_{1,j,t} \\ u_{2,j,t} \\ \cdots \\ u_{m,j,t} \end{pmatrix} \\ \cdots \\ \sum_{j=1}^{d} \gamma_j^{(d)} \begin{pmatrix} u_{1,j,t} \\ u_{2,j,t} \\ \cdots \\ u_{m,j,t} \end{pmatrix} \end{pmatrix} &= \begin{pmatrix} \sum_{i,j} u_{i,j,t}\mathbf{e}_i\gamma_j^{(1)} \\ \sum_{i,j} u_{i,j,t}\mathbf{e}_i\gamma_j^{(2)} \\ \cdots \\ \sum_{i,j} u_{i,j,t}\mathbf{e}_i\gamma_j^{(d)} \end{pmatrix} \\ &= \begin{pmatrix} \mathbf{u}_t^{(1)} \\ \mathbf{u}_t^{(2)} \\ \cdots \\ \mathbf{u}_t^{(d)} \end{pmatrix} \\ &= (\mathbf{v}_1^{(1)}, ..., \mathbf{v}_t^{(d)})^T. \end{aligned}$$

□

We now wish to encode the conjugates $Q^{(i)}$ and the $\tau^{(i)}$ into a matrix of size mnd. In order to do this we construct the block matrix $Z_0(\tau) = (Z_{i,j}(\tau)) \in M_{mnd}(\mathbb{C})$, where for each $1 \leq i, j \leq n$ we have $Z_{i,j}(\tau) = \mathrm{diag}(\tau_{i,j}^{(1)} Q^{(1)}, ..., \tau_{i,j}^{(d)} Q^{(d)}) \in M_{md}(\mathbb{C})$.

Letting $Z_1(\tau) = G'^T Z_0(\tau) G'$ (which is in $\mathcal{H}_{mnd}$) we can now prove the relation mentioned earlier.

Lemma 0.2. *For any $U \in V$ we have $Q_{V,\mathbb{Q}}(U\tau) = \phi'(U)^T Z_1(\tau) \phi'(U)$.*

Proof. It is clear by the lemma that

$$\begin{aligned}
\phi'(U)^T Z_1(\tau)\phi'(U) &= (G'\phi'(U))^T Z_0(\tau)(G'\phi'(U)) \\
&= \sum_{i,j} (\mathbf{v}_i^{(1)}, \mathbf{v}_i^{(2)}, ..., \mathbf{v}_i^{(d)}) Z_{i,j}(\tau) (\mathbf{v}_j^{(1)}, \mathbf{v}_j^{(2)}, ..., \mathbf{v}_j^{(d)})^T \\
&= \sum_{i,j} (\mathbf{v}_i^{(1)}, \mathbf{v}_i^{(2)}, ..., \mathbf{v}_i^{(d)}) \begin{pmatrix} \tau_{i,j}^{(1)} Q^{(1)} \mathbf{u}_j^{(1)} \\ \tau_{i,j}^{(2)} Q^{(2)} \mathbf{u}_j^{(2)} \\ ... \\ \tau_{i,j}^{(d)} Q^{(d)} \mathbf{u}_j^{(d)} \end{pmatrix} \\
&= \sum_{t=1}^{d} \left(\sum_{i,j} \mathbf{v}_i^{(t)} Q^{(t)} \mathbf{u}_j^{(t)} \tau_{i,j}^{(t)} \right) \\
&= \sum_{t=1}^{d} \left(\sum_{i,j} Q[U]_{i,j}^{(t)} \tau_{i,j}^{(t)} \right) \\
&= \sum_{t=1}^{d} (\sigma(Q[U]^{(t)} \tau^{(t)})) \\
&= \sigma \left(\sum_{t=1}^{d} Q[U]^{(t)} \tau^{(t)} \right) \\
&= Q_{V,\mathbb{Q}}(U\tau).
\end{aligned}$$

In the third to last equality we use the identity $\sigma(A^T B) = \sum_{i,j} A_{i,j} B_{i,j}$ with the symmetry of $Q[U]^{(t)}$. □

One also proves in a similar fashion that $B_{V,\mathbb{Q}}(U, W) = \phi'(U) Z_1(I) \phi'(W)$, a fact we will need later.

Now consider an $\mathcal{O}_K$-lattice $\mathcal{L} \subseteq V$ of rank mn. We do not assume that $\mathcal{L}$ is free. However restricting scalars to $\mathbb{Z}$ we must obtain a free $\mathbb{Z}$-lattice of rank mnd. Let $U_1, U_2, ..., U_{mnd}$ be a $\mathbb{Z}$-basis. We can restrict the quadratic forms $Q_{V.K}$ and $Q_{V,\mathbb{Q}}$ to $\mathcal{L}$ to get a quadratic forms $Q_{\mathcal{L},K}$ and $Q_{\mathcal{L},\mathbb{Q}}$.

Let $T \in \mathrm{GL}_{mnd}(\mathbb{Q})$ be the matrix whose ith column is $\phi'(U_i)$. If $U = \sum_{r=1}^{mnd} u_r U_r \in \mathcal{L}$ then it is clear that $\phi'(U) = T\mathbf{u}$, where $\mathbf{u} = (u_1, ..., u_{mnd})^T \in \mathbb{Z}^{mnd}$. Thus by the lemma we see that $Q_{\mathcal{L},\mathbb{Q}}(U\tau) = \mathbf{u}^T Z(\tau)\mathbf{u}$, where $Z(\tau) = T^T Z_1(\tau) T$. In particular the map $\phi'' : V \to \mathbb{Q}^{mnd}$ given by $\phi''(U) = T^{-1}\phi'(U)$ gives an isometry between $Q_{\mathcal{L},\mathbb{Q}}$ and the quadratic form on $\mathbb{Z}^{mnd}$ with matrix $Z(I)$.

We will need several properties of $Z(\tau)$.

Lemma 0.3. • $-Z(\tau)^{-1} = Z(I)^{-1}Z(-\tau^{-1})Z(I)^{-1}$.

• $det(-Z(\tau))^{-\frac{1}{2}} = \frac{1}{\sqrt{det(Z(I))}} det(N(\tau))^{-\frac{m}{2}}$, *where* $N(\tau) = \prod_{t=1}^{d} \tau^{(t)}$.

Consider the dual of $\mathcal{L}$ with respect to $Q_{V,K}$, i.e. $\mathcal{L}^* = \{W \in V \mid B_{V,K}(U,W) \in \mathcal{O}_K, \forall U \in \mathcal{L}\}$. The compliment of $\mathcal{L}$ is defined to be $\mathcal{L}^\# = \mathfrak{d}^{-1}\mathcal{L}^*$ where $\mathfrak{d}$ is the different of K. This is the maximum $\mathcal{O}_K$-lattice satisfying $\mathrm{tr}(B_{V,K}(U,W)) \in \mathbb{Z}$ for all $U \in \mathcal{L}$ and hence is the dual of $\mathcal{L}$ with respect to $Q_{V,\mathbb{Q}}$. Thus the Gram matrix of $\mathcal{L}^\#$ with respect to the above basis is $Z(I)^{-1} = (B_{V,\mathbb{Q}}(U_i,U_j))_{i,j})^{-1}$.

Recall that to $\mathcal{L}$ we have attached a theta series:

$$\theta(\mathcal{L},\tau) = \sum_{U \in \mathcal{L}} e^{\pi i Q_{\mathcal{L},\mathbb{Q}}(U\tau)}.$$

Given $W \in \mathcal{L}^\#$ we may also define the shifted theta series:

$$\theta(\mathcal{L},W,\tau) = \sum_{U \in \mathcal{L}} e^{\pi i Q_{V,\mathbb{Q}}((U+W)\tau)}.$$

We are now able to prove the following inversion formula for $\theta(\mathcal{L},W,\tau)$.

Theorem 0.4. *(Inversion formula) If* $W \in \mathcal{L}^\#$ *then:*

$$\theta(\mathcal{L},W,\tau) = \frac{i^{-\frac{mnd}{2}}}{\sqrt{\det(Z(I))}} \det(N(\tau))^{-\frac{m}{2}} \sum_{Y \in \mathcal{L}^\#} e^{\pi i(2B_{V,\mathbb{Q}}(W,Y) - Q_{V,\mathbb{Q}}(Y\tau^{-1}))}.$$

Proof. Let $\phi''(W) = \mathbf{w}_0$. Then using the above lemma and discussion:

$$\theta(\mathcal{L},W,\tau) = \sum_{\mathbf{u} \in \mathbb{Z}^{mnd}} e^{\pi i((\mathbf{u}+\mathbf{w}_0)^T Z(\tau)(\mathbf{u}+\mathbf{w}_0))}.$$

Since $\mathbf{w}_0 \in \mathbb{Q}^{mnd}$ is fixed and $Z(\tau) \in \mathcal{H}_{mnd}$ the right hand side is a generalised theta series with variable $Z(\tau)$. The inversion formula for such forms is known and applying this gives:

$$\begin{aligned}
\theta(\mathcal{L},W,\tau) &= \det(-iZ(\tau))^{-\frac{1}{2}} \sum_{\mathbf{u} \in \mathbb{Z}^{mnd}} e^{\pi i(-\mathbf{u}^T Z(\tau)^{-1}\mathbf{u} - 2\mathbf{u}^T\mathbf{w}_0)} \\
&= \det(-iZ(\tau))^{-\frac{1}{2}} \sum_{\mathbf{u} \in \mathbb{Z}^{mnd}} e^{\pi i((Z(I)^{-1}\mathbf{u})^T Z(-\tau^{-1})(Z(I)^{-1}\mathbf{u}) - 2\mathbf{u}^T\mathbf{w}_0)} \\
&= \det(-iZ(\tau))^{-\frac{1}{2}} \sum_{\mathbf{w} \in Z(I)^{-1}\mathbb{Z}^{mnd}} e^{\pi i(\mathbf{w}^T Z(-\tau^{-1})\mathbf{w} - 2\mathbf{w}^T Z(I)\mathbf{w}_0)} \\
&= \det(-iZ(\tau))^{-\frac{1}{2}} \sum_{\mathbf{w} \in Z(I)^{-1}\mathbb{Z}^{mnd}} e^{\pi i(2\mathbf{w}_0^T Z(I)\mathbf{w} - \mathbf{w}^T Z(\tau^{-1})\mathbf{w})} \\
&= \frac{i^{-\frac{mnd}{2}}}{\sqrt{\det(Z(I))}} \det(N(\tau))^{-\frac{m}{2}} \sum_{Y \in \mathcal{L}^\#} e^{\pi i(2B_{V,\mathbb{Q}}(W,Y) - Q_{V,\mathbb{Q}}(Y\tau^{-1}))}
\end{aligned}$$

□