An Eisenstein congruence for genus 2 Hilbert-Siegel forms.

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Joint work with D. Yhee

Outline of talk

1. Harder's conjecture
2. Ibukiyama's correspondence
3. Computations

Let $\Delta \in S_{12}(\mathrm{SL}_2(\mathbb{Z}))$ be the discriminant function. It is a normalized Hecke eigenform with Hecke eigenvalues $\tau(n)$ (the Ramanujan tau function).

Since $\mathrm{ord}_{691}\left(\frac{\zeta(12)}{\pi^{12}}\right) > 0$ we observe a congruence with the weight 12 Eisenstein series:

Ramanujan

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More "Eisenstein congruences" predicted to occur. For example genus 2 Siegel modular forms:

Harder's conjecture

Let $j > 0, k \geq 3$ and let $f \in S_{j+2k-2}(\mathrm{SL}_2(\mathbb{Z}))$ be a normalized eigenform. Suppose $\mathrm{ord}_\lambda\left(\frac{L(f,j+k)}{\Omega}\right) > 0$ for some "large prime" λ (here Ω is a canonical period).

Then there exists an eigenform $F \in S_{j,k}(\mathrm{Sp}_4(\mathbb{Z}))$ such that for all primes p:

$$a_{F,p} \equiv a_{f,p} + p^{j+k-1} + p^{k-2} \bmod \Lambda$$

where $\Lambda | \lambda$ in $\mathbb{Q}_F\mathbb{Q}_f$.

Let $K = \mathbb{Q}(\sqrt{5})$.

Conjecture

Let $j > 0, k \geq 3$ and let $f \in S_{j+2k-2}(\mathrm{SL}_2(\mathcal{O}_K))$ be a normalized eigenform. Suppose $\mathrm{ord}_\lambda\left(\frac{L(f,j+k)}{\Omega}\right) > 0$ for some "large prime" λ (here Ω is a canonical period).

Then there exists an eigenform $F \in S_{j,k}(\mathrm{Sp}_4(\mathcal{O}_K))$ such that for all prime ideals $\mathfrak{p}$:

$$a_{F,\mathfrak{p}} \equiv a_{f,\mathfrak{p}} + N(\mathfrak{p})^{j+k-1} + N(\mathfrak{p})^{k-2} \bmod \Lambda$$

where $\Lambda | \lambda$ in $\mathbb{Q}_F\mathbb{Q}_f$.

Fact

There exists an eigenform $f \in S_{12}(\mathrm{SL}_2(\mathcal{O}_K))$ such that the absolute norm of $\frac{L(f,8)}{\Omega}$ is divisible by the "large primes" $p_1 = 2687$ and $p_2 = 29203541$.

So there ought to exist eigenforms $F_1, F_2 \in S_{2,6}(\mathrm{SL}_2(\mathcal{O}_K))$ satisfying the congruences $a_{F_i,\mathfrak{p}} \equiv a_{f,\mathfrak{p}} + N(\mathfrak{p})^7 + N(\mathfrak{p})^4 \bmod \lambda_i$ where $\lambda_i \mid p_i$ in $\mathbb{Q}_{F_i}\mathbb{Q}_f$.

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Question: How to get the eigenvalues $a_{F_i,\mathfrak{p}}$?

Ibukiyama's conjecture

There exists a Hecke-equivariant isomorphism:

$$S_{2,6}(\mathrm{Sp}_4(\mathcal{O}_K)) \cong \mathcal{A},$$

where $\mathcal{A}$ is a suitable space of genus 2 algebraic modular forms of level 1.

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Let D/K be a quaternion algebra with discriminant $\infty_1\infty_2$ and consider the reductive group G/K such that:

$$G(F) = \{A \in M_2(D \otimes_{\mathbb{Q}} F) \mid A\overline{A}^T = \mu I, \mu \in (K \otimes_{\mathbb{Q}} F)^\times\},$$

for any $\mathbb{Q}$-algebra F.

Facts

- G is an inner form of $\mathrm{Res}_{K/\mathbb{Q}}(\mathrm{GSp}_4)$.
- $G(\mathbb{R})/Z(G(\mathbb{R})) \cong (\mathrm{USp}(4)/\{\pm I\})^2$ is compact.
- For p we have $G(\mathbb{Q}_q) \cong \mathrm{Res}_{K/\mathbb{Q}}(\mathrm{GSp}_4(\mathbb{Q}_q))$.

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Let V be the irrep of USp(4) with Young diagram parameters $(5,3)$ (natural description: "pluriharmonic polynomials").

By embedding into $G(\mathbb{R})$ it is clear that $G(\mathbb{Q})$ acts on $W = V \otimes V$ via $A \cdot (v_1 \otimes v_2) = (A \cdot v_1 \otimes \sigma(A) \cdot v_2)$ (sigma is the non-trivial automorphism of K).

Let $K_f = \prod_p G(\mathbb{Z}_p)$.

$\mathcal{A}$ is the space of functions $f : G(\mathbb{A}_f) \to W$ satisfying

$$f(\gamma g k) = \gamma \cdot f(g)$$

for all $(\gamma, g, k) \in G(\mathbb{Q}) \times G(\mathbb{A}_f) \times K_f$

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Hecke operators

Each $u \in G(\mathbb{A}_f)$ defines a Hecke operator T_u via:

$$T_u(f)(g) = \sum_{i=1}^{r} f(gu_i),$$

where $K_f u K_f = \coprod_{i=1}^{r} u_i K_f$.

Ibukiyama predicts:

$$T_{\mathfrak{p}} \longleftrightarrow T_{u,\mathfrak{p}}$$

where u has identity component away from $\mathfrak{p}$ and $u_{\mathfrak{p}} \mapsto \mathrm{diag}(1, 1, \pi_{\mathfrak{q}}, \pi_{\mathfrak{q}})$, depending on a choice of isomorphism $G(\mathbb{Q}_q) \cong \mathrm{GSp}_4(\mathbb{Q}_q)$ and a totally positive generator $\pi_{\mathfrak{p}}$ for $\mathfrak{p}$.

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Fact

$$G(\mathbb{A}_f) = G(\mathbb{Q})K_f.$$

So every $f \in \mathcal{A}$ is determined by $f(\mathrm{id}) \in W$. There are further restrictions:

Lemma

The map $f \mapsto f(\mathrm{id})$ defines an isomorphism:

$$\mathcal{A} \cong W^{\Gamma},$$

where $\Gamma = G(\mathbb{Q}) \cap K_f = G(\mathbb{Z})$, a finite group of size 28800.

So $\dim(W^{\Gamma}) = \sum_{\gamma \in \Gamma} \chi(\gamma) = 3$ by the Weyl character formula.

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How to compute Hecke representatives for $T_{u,\mathfrak{p}}$?

By local considerations we find that:

Let $G(\mathbb{Q})_{\mathfrak{p}}$ be the subgroup of similitude $\pi_{\mathfrak{p}}$ matrices and $Y_{\mathfrak{p}} = G(\mathbb{Q})_{\mathfrak{p}} \cap M_2(\mathcal{O}_D)^{\times}$. Then:

$$K_f u K_f = \coprod_{[u_i] \in Y_{\mathfrak{p}}/\Gamma} u_i K_f.$$

$$\deg(T_{u,\mathfrak{p}}) = (N(\mathfrak{p})+1)(N(\mathfrak{p})^2+1).$$

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Although $\mathcal{A} \cong W^\Gamma$ is 3-dimensional it is non-trivial to compute a basis since $\dim(W) = 154^2 = 23716$.

Strategy:

1. Find a basis $f_1, ..., f_{154}$ for V, hence a basis for W.
2. Run through the elements $w_{i,j} = f_i \otimes f_j \in W$ and compute:

$$R(w_{i,j}) = \frac{1}{28800} \sum_{\gamma \in \Gamma} \gamma \cdot w_{i,j}.$$

   It is clear that $R(w_{i,j}) \in W^\Gamma$, but it is almost always 0.
3. Stop once you find 3 linearly independent $R(w_{i,j})$.

This takes a long time to run, there seems to be no obvious way to determine which of the $w_{i,j}$ give non-zero output.

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The congruences are now in sight. We will only see T_2 today. It takes a few minutes to compute the $(4+1)(16+1) = 85$ Hecke representatives for $T_{u,2}$.

The matrix of $T_{u,2}$ takes much longer to compute:

$$\begin{pmatrix} 640 & \frac{10368\sqrt{5}}{5} & \frac{-1984\sqrt{5}}{111} \\ \frac{575\sqrt{5}}{16} & 460 & \frac{1750}{111} \\ \frac{4995\sqrt{5}}{8} & 0 & 1360 \end{pmatrix}$$

Thus the eigenvalues $\alpha_1, \alpha_2, \alpha_3$ give the Hecke eigenvalues $b_{F,2}$ for the three eigenforms $F \in S_{2,6}(\mathrm{Sp}_4(\mathcal{O}_K))$.

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The Hilbert modular form $f \in S_{12}(\mathrm{SL}_2(\mathcal{O}_K))$ mentioned earlier has eigenvalue $a_{f,2} = 170 + 30\sqrt{809}$.

Let L be the compositum of $\mathbb{Q}(\sqrt{809})$ and the splitting field of h. Magma tells us that one of the values

$$N_{L/\mathbb{Q}}(\alpha_i - (170 + 30\sqrt{809} + 4^7 + 4^4))$$

is divisible by 2687 and another by 29203541, as expected!

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Thanks for listening!