# MATH7421 Useful Information Sheet
# (Algebraic Number Theory)

## Polynomial Irreducibility Tests

1. **Extension Field Test:** Let $f(x) \in K[x]$ with $\deg(f) \geq 1$ and $L$ be a field containing $K$. If $f$ is irreducible over $L$ then $f$ is irreducible over $K$.
2. **Rational root test:** Let $f(x) = a_0 + a_1x + a_2x^2 + ... + a_nx^n \in \mathbb{Z}[x]$ have $\deg(f(x)) \geq 1$ and $a_0 \neq 0$. If $f(a) = 0$ for some $a = \frac{b}{c} \in \mathbb{Q}$ then $b|a_0$ and $c|a_n$.
3. **Gauss' Lemma** If $f(x) \in \mathbb{Z}[x]$ is reducible over $\mathbb{Q}$ then there exists a non-trivial factorisation with factors in $\mathbb{Z}[x]$.
4. **Eisenstein's Criterion** Let $f(x) = a_0 + a_1x + a_2x^2 + ... + a_nx^n \in \mathbb{Z}[x]$ and suppose that there is a prime $p$ such that:
   - $p \nmid a_n$
   - $p|a_i$ for each $0 \leq i \leq n-1$
   - $p^2 \nmid a_0$

   Then $f(x)$ is irreducible over $\mathbb{Q}$.

## Discriminant of an $n$-tuple of a number field

1. **Definition:** Let $\mathbb{Q}(\alpha)$ be a number field of degree $d$ and $\beta_1, \beta_2, ..., \beta_d \in \mathbb{Q}(\alpha)$. The **discriminant** is $\Delta(\beta_1, \beta_2, ..., \beta_d) = \det(M_{\beta_1,\beta_2,...,\beta_d})^2$, where:
$$M_{\beta_1,\beta_2,...,\beta_d} = \begin{pmatrix} \sigma_1(\beta_1) & \sigma_1(\beta_2) & \dots & \sigma_1(\beta_d) \\ \sigma_2(\beta_1) & \sigma_2(\beta_2) & \dots & \sigma_2(\beta_d) \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_d(\beta_1) & \sigma_d(\beta_2) & \dots & \sigma_d(\beta_d) \end{pmatrix}$$
2. **Properties:** Let $\mathbb{Q}(\alpha)$ be a number field of degree $d$ and $\beta_1, \beta_2, ..., \beta_d \in \mathbb{Q}(\alpha)$ be a basis.
   - $\Delta(\beta_1, \beta_2, ..., \beta_d) \in \mathbb{Q}\backslash\{0\}$.
   - If $\beta'_j = \sum_{i=1}^d c_{i,j}\beta_i$ for some $c_{i,j} \in \mathbb{Q}$ then
   $$\Delta(\beta'_1, \beta'_2, ..., \beta'_d) = (\det(C))^2\Delta(\beta_1, \beta_2, ..., \beta_d)$$
   where $C$ is the matrix having $(i,j)$-entry $c_{i,j}$.
   - If $\gamma_1, \gamma_2, ..., \gamma_d \in \mathcal{O}_K$ are a basis for $K$ then $\Delta(\gamma_1, \gamma_2, ..., \gamma_d) \in \mathbb{Z}\backslash\{0\}$.

**Noetherian Criteria**

1. An integral domain is Noetherian if it satisfies one of the following equivalent conditions:

    - Every non-empty set $S$ of ideals of $R$ has a maximal element with respect to inclusion, i.e. there exists $I \in S$ that is not contained in any other $J \in S$.
    - Every ideal of $R$ is **finitely generated**.
    - Every infinite sequence of ideals $I_1, I_2, I_3...$ of $R$ such that:

    $$I_1 \subseteq I_2 \subseteq I_3 \subseteq ...$$

    eventually stabilises, i.e. at some point we have $I_n = I_{n+1} = I_{n+2} = ...$