MAS430 - Chapter 1 Exercises

Infinitude of primes

1. Suppose there are finitely many distinct primes $p_1, p_2, ..., p_n$. In proving the infinitude of primes we usually construct the number:

$$N = p_1 p_2 ... p_n + 1.$$

(Note that the two terms in this sum have product $P = p_1 p_2 ... p_n$)

Show that the proof still works if we consider **any** factorisation $P = mn$ and construct $N = m + n$.

2. Given $a \geq 2$ with at least r prime factors, show that $a(a+1)$ must have **at least** $r+1$ prime factors. From this deduce that there are infinitely many primes.

3. (a) Let $f(x) = x^2 + x + 41$. Verify that $f(n)$ is prime for $n = 0, 1, 2, ..., 39$. **Without** working out the values of $f(40)$ and $f(41)$ show that these two values are definitely not prime.

   (b) Let $f(x) \in \mathbb{Z}[x]$ be a non-constant polynomial. Show that $f(n)$ cannot be prime for all n.

   (Hint: Assume so and let $p = f(1)$. Try to show that the polynomial $g(x) = f(x) - p$ has infinitely many roots).

4. We consider another "analytic" proof of the infinitude of primes. Let $N \geq 2$.

   (a) Show that for any $n \in \mathbb{N}$ the number of positive integers less than or equal to N that are divisible by n^2 is $\lfloor \frac{N}{n^2} \rfloor$.

   (b) Recall that an integer $m \geq 2$ is **square-free** if it has a factorisation of the form $m = p_1 p_2 ... p_k$ for **distinct** primes $p_1, p_2, ..., p_k$ (i.e. m isn't divisible by any square number except 1).

   Define for $x \geq 2$ the function:

$$A(x) = \#\{m \leq x \mid m \text{ is square-free}\}.$$

   Show that $N - A(N) \leq \sum_{n=2}^{\infty} \frac{N}{n^2}$.

(c) Using the fact that $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$ (to be proved later in the course), show that:

$$A(N) \geq N\left(2 - \frac{\pi^2}{6}\right)$$

and hence prove that there are infinitely many primes.

Primes in arithmetic progressions

1. Using standard techniques or otherwise, prove that there are infinitely many primes of the form $6k - 1$.
2. Let q be a fixed prime. We will show that there are infinitely many primes congruent to 1 mod q.

   Consider the qth **cyclotomic polynomial**:

   $$\Phi_q(x) = x^{q-1} + x^{q-2} + ... + x + 1.$$

   (a) Show that among the list of numbers $\Phi_q(1), \Phi_q(2), \Phi_q(3), ...$ there are infinitely many prime divisors.

   (b) Suppose that $p > q$ is a prime such that $p \mid \Phi_q(a)$ for some $a \in \mathbb{Z}$. Show that $a^q \equiv 1 \bmod p$ and deduce from this that $\overline{a} \in (\mathbb{Z}/p\mathbb{Z})^{\times}$ has order q.

   (c) For such a p show that $p \equiv 1 \bmod q$ and so deduce that there are infinitely many primes congruent to 1 mod q.
3. Let $L \geq 1$ be any number ending in $1, 3, 7$ or 9. Assuming Dirichlet's theorem show that there are infinitely many primes ending with the string of digits given by L.

Bertrand's postulate and Chebyshev's inequalities

1. Let $n \geq 2$ and p be prime.

   (a) Show that the exact power of p dividing $n!$ is:

   $$\sum_{k=1}^{\infty} \left\lfloor \frac{n}{p^k} \right\rfloor.$$

   Is this **really** an infinite sum?

   (b) Prove the claim in the proof of Lemma 1.10, that the exact power of p dividing $\binom{2n}{n}$ is p^{α_p} where:

   $$\alpha_p = \sum_{k=1}^{r_p} \left(\left\lfloor \frac{2n}{p^k} \right\rfloor - 2 \left\lfloor \frac{n}{p^k} \right\rfloor \right),$$

   (recall that r_p is such that $p^{r_p} \leq 2n < p^{r_p+1}$).

2. In this exercise we prove Lemma 1.12, that:

$$\prod_{p \leq n} p < 4^n.$$

   (a) Show that it suffices to prove the result for odd n.
   (b) Show that for $n = 2m+1$ we have:

$$\binom{2m+1}{m} \leq 4^m.$$

   Hint: use the binomial expansion of $(1+1)^{2m+1}$.
   (c) Using strong induction prove the Lemma.
   Hint: For $n = 2m+1$ odd consider splitting the product as follows:

$$\prod_{p \leq 2m+1} p = \left(\prod_{p \leq m+1} p\right)\left(\prod_{m+1 < p \leq 2m+1} p\right).$$

3. Show that there are infinitely many primes beginning with the digit 1.

4. Let $n \geq 2$. Using Bertrand's postulate:

   (a) Show that $\sum_{k=1}^{n} \frac{1}{k}$ is not an integer.
   (b) Show that $n!$ is not a square.

Asymptotics and the prime number theorem

1. Prove that the relation of "being asymptotic to" is an equivalence relation on the set of positive real functions (for simplicity).

2. Show the following:

   (a) $x^3 + 4x + 5 \sim x^3$,
   (b) If $P(x) \in \mathbb{R}[x]$ has leading term $\alpha_n x^n$ then $P(x) \sim \alpha_n x^n$.
   (c) $\sqrt{x^2 + A} \sim x$.
   (d) $\frac{1}{x+A} \sim \frac{1}{x}$ (for $x > -A$).

3. Let f be a positive real function such that $f(x) \to \infty$ as $x \to \infty$.

   (a) Show that for any constant $A > 0$ we have $f(x) + A \sim f(x)$.
   (b) Show that for any bounded function $g(x)$ we have $f(x)+g(x) \sim f(x)$.
   (c) Give an example to show that the above is not necessarily true if $g(x)$ is unbounded.

4. Let f, g, h be positive real functions and fix a positive constant A. Show that if $f \sim h$ and $g \sim h$ then $f + g \sim 2h$ and $Af \sim Ag$.

5. Let $m \geq 2$. By using the previous two exercises and the PNT prove the second claim of Theorem 1.30:

$$\pi_{m,a}(x) \sim \frac{x}{\phi(m)\ln(x)}$$

by assuming the first claim holds, i.e. that $\pi_{m,a}(x) \sim \pi_{m,b}(x)$ for any a, b coprime to m.

6. Using integration by parts prove that:

$$\frac{x}{\ln(x)} \sim \int_2^x \frac{dt}{\ln(t)}.$$

(as mentioned in "Interesting Fact - A brief history of the PNT").

7. If p_n is the nth prime then show:

$$\lim_{n\to\infty} \frac{p_n}{p_{n+1}} = 1.$$

8. Let $0 < a < b$.

   (a) Using the PNT show that:

   $$\lim_{x\to\infty} \frac{\pi(bx)}{\pi(ax)} = \frac{b}{a}.$$

   (b) Deduce that for x large enough there is always a prime between bx and ax.

   (c) Let $L \geq 1$ be any number. Show that there are infinitely many primes beginning with the string of digits given by L. (Hint: Let $a = L$ and $b = L + 1$ in the above. Make a suitable choice for x.

   (d) Show that there are primes p, q such that $a < \frac{p}{q} < b$. (Hint: again use the above for a suitable choice of x).

MAS430 - Chapter 2 Exercises

Arithmetic functions

1. Prove the following from Example 2.4 and demonstrate each for input $n = 10$:
   (a) $I = \mu \star u$
   (b) $\ln = \Lambda \star u$
   (c) $N = \phi \star u$

2. (a) Prove that the set of arithmetic functions with $f(1) \neq 0$ is a group by checking the axioms explicitly (you may assume that $f(1) \neq 0$ implies that the Dirichlet inverse exists).
   (b) Consider the subset of arithmetic functions with $f(1) = 1$. Is this a subgroup?

3. Show that for the group in 2(a) the 2-torsion (i.e. elements of order 1 or 2) consists of $\pm I$.

4. Show that the only arithmetic functions to satisfy $f \star f = f$ are $f = \mathbf{0}, I$.

5. Show, using known Dirichlet convolutions, that $\sigma = \phi\star\sigma_0$ and demonstrate for $n = 10$.

6. (a) Show that ϕ, σ_α and μ are multiplicative functions.
   (b) Show the following formula for σ_α if $n = p_1^{k_1} p_2^{k_2} ... p_m^{k_m}$ and $\alpha \neq 0$:
   $$\sigma_\alpha(n) = \prod_{i=1}^{m} \frac{p_i^{(k_i+1)\alpha} - 1}{p_i^\alpha - 1}.$$
   (c) Using only the factorisation of n (and not multiplicativity) argue that:
   $$\sigma_0(n) = \prod_{i=1}^{m} (1 + k_i).$$
   Does this agree with the formula above?

7. Let f be multiplicative. Prove the claim in Example 2.9, that if f is completely multiplicative then the Dirichlet inverse of f is μf. Is the converse true?

8. Prove the following identity and demonstrate it for $n = 20$:

$$n = \sum_{d|n} \sigma(d)\mu\left(\frac{n}{d}\right).$$

(Hint: $\sigma = N \star u$).

9. Prove the following identity and demonstrate it for $n = 20$ and $n = 81$:

$$\Lambda(n) = \sum_{d|n} \ln(d)\mu\left(\frac{n}{d}\right).$$

Dirichlet series and Euler products

1. Show that $D(s, N_\alpha) = \zeta(s - \alpha)$ and thus prove the formal identity:

$$\sum_{n=1}^{\infty} \frac{\sigma(n)}{n^s} = \zeta(s)\zeta(s-1).$$

2. (a) Express $D(s, \sigma_\alpha)$ in terms of the Riemann zeta function (in a similar fashion to the above exercise).
   (b) What do you notice about the result for $D(s, \sigma_0)$.
   (c) Assuming Exercise 5 on arithmetic functions show that:

$$D(s, \phi) = \frac{\zeta(s-1)}{\zeta(s)}.$$

   Does this match what you would get if you instead used the convolution $N = \phi \star u$?

3. Recall that μ is multiplicative (but not completely multiplicative). By writing down the Euler product expansion of $D(s, \mu)$ find an alternative proof that:

$$D(s, \mu) = \frac{1}{\zeta(s)}.$$

4. Let f be completely multiplicative. Then using Exercise 7 on arithmetic functions find an alternative proof of the Euler product epansion. (You may assume the Euler product expansion for multiplicative functions or use an alternative proof).

5. Let P be the arithmetic function defined by:

$$P(n) = \begin{cases} 0 & \text{if } n = 1 \\ k & \text{if } n = p_1^{a_1} p_2^{a_2} ... p_k^{a_k} \end{cases}$$

   (a) Show that if m, n are coprime then $P(mn) = P(m) + P(n)$ (such functions are called additive). Deduce that the arithmetic function $2^{P(n)}$ is multiplicative.

(b) Show that:

$$D(s, 2^{P(n)}) = \frac{\zeta(s)^2}{\zeta(2s)}.$$

(c) Using this deduce that:

$$\sum_{\text{hcf}(a,b)=1} \frac{1}{a^2b^2} = \frac{\zeta(2)^2}{\zeta(4)}.$$

(Later we will find the value of the RHS, it will be $\frac{5}{2}$).

6. (a) Using the geometric sum formula show the identity:

$$1 + 4x + 9x^2 + 16x^3 + ... = \frac{1-x^2}{(1-x)^4}.$$

(b) By considering Euler products show that:

$$D(s, \sigma_0^2) = \frac{\zeta(s)^4}{\zeta(2s)}.$$

7. (a) Let $S_k(x) = 1 + x + x^2 + ... + x^k$. Prove for any $\alpha, \beta \neq 1$ the identity:

$$\sum_{n=0}^{\infty} S_n(\alpha)S_n(\beta)x^n = \frac{(1-\alpha\beta x^2)}{(1-x)(1-\alpha x)(1-\beta x)(1-\alpha\beta x)}.$$

(b) Hence show that:

$$D(s, \sigma_a\sigma_b) = \frac{\zeta(s)\zeta(s-a)\zeta(s-b)\zeta(s-a-b)}{\zeta(2s-a-b)}.$$

8. By considering Euler products show that:

$$\frac{\zeta'(s)}{\zeta(s)} = -D(s, \Lambda)$$

(Hint: Take logs **before** differentiating).

(The above link is important in proving the PNT and demonstrates the importance of the Von-Mangoldt function Λ).

9. Let f be a multiplicative function. Suppose there exists a fixed integer $k \geq 1$ such that for all primes p and all $m \geq 2$ we have the following:

$$f(p^m) = f(p)f(p^{m-1}) - p^{k-1}f(p^{m-2}).$$

(a) Is f completely multiplicative?

(b) Show that that Euler product expansion is of the following form:

$$D(s, f) = \prod_p \left(1 - f(p)p^{-s} + p^{k-1-2s}\right)^{-1}.$$

(Note that the factors in this Euler product are quadratic in p^{-s} whereas so far in this course they have been linear in p^{-s}. This opens up a new world of possibilities.

This exercise gives us the Euler product expansion for Dirichlet series attached to certain modular forms, a special class of functions arising in advanced number theory.

As a bonus you might like to prove that any Dirichlet series of the above form indeed gives a multiplicative function with the above properties.)

Convergence of Dirichlet series

1. This exercise invstigates the effect of rearranging terms in a certain infinite sum.

   Let:
   $$S = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}.$$

   (a) By using the Taylor expansion for $\ln(1+x)$ show that $S = \ln(2)$. Does S converge absolutely?

   (b) Show that we may rearrange the terms of S to form the sum:
   $$T = \sum_{n=1}^{\infty} \left(\frac{1}{2n-1} - \frac{1}{2(2n-1)} - \frac{1}{4n} \right).$$

   (c) Show that $T = \frac{\ln(2)}{2} \neq S$.
   (In general for any $a, b \in \mathbb{N}$ we may rearrange in the following way:
   $$W = \sum_{n=1}^{\infty} \left(\sum_{m=0}^{na} \frac{1}{2m-1} - \sum_{m=1}^{nb} \frac{1}{2m} \right)$$
   and it turns out that $W = \frac{\ln(4a)-\ln(b)}{2}$. The above is the case $a = 1, b = 2$.)

2. For the following you may assume that $\zeta(s)$ converges absolutely for $\mathrm{Re}(s) > 1$ (a fact we will prove in Chapter 3).

   (a) Suppose that f is an arithmetic function such that $|f(n)| \leq M$ for all n. Show that $D(s,f)$ converges absolutely for $\mathrm{Re}(s) > 1$.

   (b) Show that $D(s, N_\alpha)$ converges absolutely for $\mathrm{Re}(s) > 1 + \mathrm{Re}(\alpha)$.

   (c) Suppose that f is an arithmetic function such that $|f(n)| \leq Cn^\delta$ for some constants $C, \delta > 0$. Show that $D(s,f)$ converges absolutely for $\mathrm{Re}(s) > 1 + \delta$. Does this also imply part (b)?

3. Suppose f is a real arithmetic function taking only positive values. What can be said about the abscissae of convergence and absolute convergence of $D(s,f)$?

MAS430 - Chapter 3 Exercises

Behaviour of $\zeta(s)$.

1. In this question we prove the claim before Lemma 3.2, that $\zeta(s)$ behaves like $\frac{1}{\sigma-1}$ as $\sigma \to 1$.

   (a) Show for any $\sigma > 1$ and $n \geq 1$ that $\int_n^{n+1} \frac{dx}{x^\sigma} = \frac{n^{1-\sigma}-(n+1)^{1-\sigma}}{\sigma-1}$.

   (b) Deduce that for $\sigma > 1$ and $n \geq 1$:

   $$(n+1)^{-\sigma} < \frac{n^{1-\sigma} - (n+1)^{1-\sigma}}{\sigma - 1} < n^{-\sigma}.$$

   (c) Hence show that for $\sigma > 1$:

   $$\frac{1}{\sigma - 1} < \zeta(\sigma) < \frac{1}{\sigma - 1} + 1.$$

Bernoulli numbers/polynomials and special values of ζ.

1. We will see an alternate proof that $\zeta(2) = \frac{\pi^2}{6}$.

   (a) Let $S = [0,1] \times [0,1]$ and T be the right angled triangle passing through points $(0,0), (\frac{\pi}{2}, 0), (0, \frac{\pi}{2})$.
   Using the substitution:

   $$x = \frac{\sin u}{\cos v}, \qquad y = \frac{\sin v}{\cos u}$$

   show that:

   $$\int\int_S \frac{dx\,dy}{1-(xy)^2} = \int\int_T dudv$$

   and hence get a value for this integral.

   (b) By evaluating the integral over S a different way, show that:

   $$1 + \frac{1}{3^2} + \frac{1}{5^2} + ... = \frac{\pi^2}{8}.$$

   (c) Hence show that $\zeta(2) = \frac{\pi^2}{6}$.

2. (a) Prove the following identity for the Bernoulli numbers (for $k \geq 2$):

$$\sum_{n=0}^{k-1} \binom{k}{n} B_n = 0.$$

   (b) Use it recursively (with $B_0 = 1, B_1 = -\frac{1}{2}$ and $B_2 = \frac{1}{6}$) to find all Bernoulli numbers up to B_{12}.

3. (a) Prove the following formula for $B_k(x)$:

$$B_k(x) = \sum_{n=0}^{k} \binom{k}{n} B_n x^{k-n}.$$

   Use it to generate $B_1(x), B_2(x), B_3(x), B_4(x)$ and $B_5(x)$ and check your answers with the notes (you may use your list of Bernoulli numbers found in Exercise 2).

   (b) What happens if we let $x = 1$ in the above formula?

4. Show that for $n \geq 1$ we have $B_n(x+1) - B_n(x) = nx^{n-1}$.

5. For $k, n \geq 1$ let:

$$S_k(n) = \sum_{m=1}^{n} m^k.$$

   In this question we prove the formula:

$$S_k(n) = \frac{1}{k+1} \sum_{r=0}^{k} \binom{k+1}{r} B_r \cdot (n+1)^{k+1-r},$$

   (as mentioned in "Interesting fact - Sums of powers").

   (a) Using Exercise 4 show that:

$$S_k(n) = \frac{B_{k+1}(n+1) - B_{k+1}(1)}{k+1}$$

   and then prove the formula by using Exercise 3.

   (b) Generate formulae for $S_1(n), S_2(n), S_3(n), S_4(n)$ (some of these should look familiar). Test your formulae for $n = 4$.

6. Use your list of Bernoulli numbers in Exercise 2 to find $\zeta(4), \zeta(6), \zeta(8), \zeta(10)$ and $\zeta(12)$.

7. In this question we will study Euler's original method of finding $\zeta(2k)$.

   Recall the infinite product expansion given in the notes (we will ignore convergence issues but everything we do can be shown to be valid):

$$\frac{\sin z}{z} = \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{(\pi n)^2}\right).$$

(a) By taking logarithmic deriviatives of both sides, show that:

$$\frac{z\cos z}{\sin z} = 1 - 2\sum_{k=1}^{\infty} \frac{\zeta(2k)}{\pi^{2k}} z^{2k}.$$

(b) By writing in terms of complex exponentials show that:

$$\frac{z\cos z}{\sin z} = 1 + \sum_{k=1}^{\infty} (-1)^k \frac{B_{2k} 2^{2k}}{(2k)!} z^{2k}.$$

Hence derive the formula:

$$\zeta(2k) = (-1)^{k+1} \frac{B_{2k}(2\pi)^{2k}}{2(2k)!}.$$

8. (a) Let $k \geq 0$ be an integer. By using the Fourier expansions for $P_n(x)$ find a formula for the value of the sum:

$$S_k = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^{2k+1}} = 1 - \frac{1}{3^{2k+1}} + \frac{1}{5^{2k+1}} - ...$$

(b) Explicitly find the values of S_0, S_1 and S_2 using the formula.

MAS430 - Chapter 4 Exercises

Characters

1. (a) For $m = 3, 7, 10, 24$ find explicit descriptions for the group $(\mathbb{Z}/m\mathbb{Z})^{\times}$.
   (b) For each value of m find the character group and check that it has $\phi(m)$ elements, same as $(\mathbb{Z}/m\mathbb{Z})^{\times}$.
   (c) For each of your answers demonstrate the orthogonality relations.

2. Let χ be a mod m Dirichlet character and $\zeta_m = e^{\frac{2\pi i}{m}}$ (a primitive mth root of unity). For $a \in \mathbb{Z}$ define the **Gauss sum**:

$$G(a, \chi) = \sum_{k=1}^{m} \chi(k)\zeta_m^{ak}.$$

   (a) Calculate $G(0, \chi)$ for any χ.
   (b) For a coprime to m show that $G(a, \chi) = \overline{\chi(a)}G(1, \chi)$ for any χ. (this result is actually true even if a isn't coprime to m).
   (c) For an odd prime p let $\chi_p = \left(\frac{\cdot}{p}\right)$ be the Legendre symbol . Find the values of $G(1, \chi_3)^2, G(1, \chi_5)^2$ and $G(1, \chi_7)^2$.
   (d) In general show that $G(1, \chi_p)^2 = (-1)^{\frac{(p-1)}{2}} p$.

   (The above result (originally due to Gauss) proves that the cyclotomic field $\mathbb{Q}(\zeta_p)$ contains a quadratic subfield $\mathbb{Q}\left(\sqrt{(-1)^{\frac{p-1}{2}} p}\right)$. This leads to many interesting things such as explicit constructibility of polygons and a proof of quadratic reciprocity.
   Gauss sums themselves are very important in number theory. They appear in the functional equation for Dirichlet L-series amongst other things).

Dirichlet L-series and primes in arithmetic progressions.

1. (a) For each non-trivial character χ of $(\mathbb{Z}/10\mathbb{Z})^{\times}$ found in Exercise 1 show that $L(1, \chi) \neq 0$.

(b) Check explicitly that for a fixed prime p:

$$\sum_{\chi} \chi(3)^{-1}\chi(p) = \begin{cases} 4 & \text{if } p \equiv 3 \bmod 10 \\ 0 & \text{otherwise} \end{cases}$$

(c) Using Dirichlet's argument show that there are infinitely many primes congruent to 3 mod 10 (only a brief argument is required, you may assume the convergence properties as outlined in the notes).

2. In this question we investigate $L(1,\chi)$ for a specific χ and find an amazing relationship with Pell's equation/units of $\mathbb{Z}[\sqrt{3}]$.

Let χ_{12} be the following mod 12 Dirichlet character:

$$\chi_{12}(n) = \begin{cases} 1 & \text{if } n \equiv 1, 11 \bmod 12 \\ -1 & \text{if } n \equiv 5, 7 \\ 0 & \text{otherwise} \end{cases}$$

(a) Define the power series:

$$f_{12}(x) = \sum_{k=1}^{\infty} \left(\frac{x^{12k+1}}{12k+1} - \frac{x^{12k+5}}{12k+5} - \frac{x^{12k+7}}{12k+7} + \frac{x^{12k+11}}{12k+11} \right).$$

Show that $f_{12}'(x) = \frac{1-x^2}{x^4-x^2+1}$

(b) Using partial fractions show that $f_{12}(x) = \frac{1}{2\sqrt{3}} \ln\left(\frac{x^2+\sqrt{3}x+1}{x^2-\sqrt{3}x+1}\right) + C$ for some constant C.

(c) Hence show that $L(1,\chi_{12}) = \frac{1}{\sqrt{3}} \ln(2+\sqrt{3}) \neq 0$.

(d) Show that $2+\sqrt{3}$ is a unit of $\mathbb{Z}[\sqrt{3}]$ and deduce from this that $(2,1)$ is a solution of the Pell equation $x^2 - 3y^2 = 1$
(You may recall that $(2,1)$ is the "smallest" non-trivial solution to this Pell equation, as found by using continued fractions in MAS208/330. Alternatively in MAS276 you found that $(2+\sqrt{3})$ is the "smallest" unit of $\mathbb{Z}[\sqrt{3}]$ bigger than 1. This proved that the units of $\mathbb{Z}[\sqrt{3}]$ form an infinite group).

3. Consider the mod 8 Dirichlet character:

$$\chi_8(n) = \begin{cases} 1 & \text{if } n \equiv 1, 7 \bmod 8 \\ -1 & \text{if } n \equiv 3, 5 \bmod 8 \\ 0 & \text{otherwise} \end{cases}$$

Mimic the previous exercise to show that $L(1,\chi_8) = \frac{1}{2\sqrt{2}} \ln(3+2\sqrt{2}) \neq 0$. Show that $3+2\sqrt{2}$ is a unit of $\mathbb{Z}[\sqrt{2}]$ and deduce from this that $(3,2)$ is a solution of the Pell equation $x^2 - 2y^2 = 1$.

(The two characters in Exercises 2 and 3 are generalisations of the Legendre symbol called the Kronecker symbol.

Dirichlet managed to prove that for the Kronecker symbol $\chi_d = \left(\frac{d}{\cdot}\right)$ and for **certain** $d > 0$ we have:

$$L(1,\chi_d) = \frac{h}{\sqrt{d}} \ln(\epsilon),$$

where remarkably:

- ϵ is a fundamental unit of $\mathbb{Z}[\sqrt{d}]$ (giving rise to a fundamental solution of Pell's equation $x^2 - dy^2 = 1$).
- $h \geq 1$ is an important number attached to $\mathbb{Z}[\sqrt{d}]$ called the class number, measuring whether or not the ring has unique factorisation.

He went much further to invent a generalisation of this formula for general number fields called the **analytic class number formula**).