

Week 2

Polynomials 1

A polynomial is an expression of the form $a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$ for numbers $a_0, a_1, \dots, a_n$ (ie. expression featuring only positive powers of x and numerical coefficients)

The degree of a polynomial is the largest $n \geq 0$ such that $a_n \neq 0$.

Degree 1 — $ax + b$ linear

Degree 2 — $ax^2 + bx + c$ quadratic

Degree 3 — $ax^3 + bx^2 + cx + d$ cubic

⋮
We've solved some linear equations, let's move up!

Quadratics

There are many ways to solve $ax^2+bx+c=0$

① Factorization

If we can write as $(mx+n)(m'x+n')=0$

then either $mx+n=0$ or $m'x+n'=0$

$$\rightarrow x = \frac{-n}{m} \text{ or } \frac{-n'}{m'}$$

(so get up to two solutions)

Ex 1.1 $x^2+7x+10 = (x+2)(x+5) = 0$

$$\rightarrow x = -2 \text{ or } -5$$

Ex 1.2 • $x^2-6x+9 = (x-3)^2 = 0$

$$\rightarrow x = 3$$

• $x^2+4x-12 = (x+6)(x-2) = 0$

$$\rightarrow x = -6 \text{ or } 2$$

• $x^2-4 = (x-2)(x+2) = 0$

$$\rightarrow x = \pm 2$$

(could have just done $x^2=4 \rightarrow x = \pm\sqrt{4}$
here) $\stackrel{= \pm 2}{\text{here}}$

② Completing the square

Eq^s like $x^2 = t$ are easy $\rightarrow x = \pm\sqrt{t}$

It turns out that every quadratic can be put in this form ... you halve the x coefficient and subtract its square.

Ex. 1.3

$$\begin{aligned}x^2 + 4x - 5 &= (x+2)^2 - 4 - 5 \\&= (x+2)^2 - 9 = 0\end{aligned}$$

$$\rightarrow x+2 = \pm 3 \rightarrow x = -5 \text{ or } 1$$

Ex. 1.5 • $x^2 + 2x - 8 = (x+1)^2 - 9 = 0$

$$\rightarrow x+1 = \pm 3 \rightarrow x = -4, 2$$

$$\begin{aligned}4x^2 - 12x - 7 &= 4\left[x^2 - 3x - \frac{7}{4}\right] = 4\left[\left(x - \frac{3}{2}\right)^2 - \frac{9}{4} - \frac{7}{4}\right] \\&= 4\left[\left(x - \frac{3}{2}\right)^2 - 4\right] = 0\end{aligned}$$

$$\rightarrow x - \frac{3}{2} = \pm 2 \rightarrow x = -\frac{1}{2} \text{ or } \frac{7}{2}$$

③ Quadratic Formula

Let's complete the square in general!

Ex 1.4 $ax^2 + bx + c = a\left[x^2 + \frac{b}{a}x + \frac{c}{a}\right]$

$$\begin{aligned}&= a\left[\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a^2} + \frac{c}{a}\right] \\&= a\left[\left(x + \frac{b}{2a}\right)^2 - \left(\frac{b^2 - 4ac}{4a^2}\right)\right]\end{aligned}$$

So $ax^2 + bx + c = 0$

$$\rightarrow x + \frac{b}{2a} = \pm \sqrt{\frac{b^2 - 4ac}{4a^2}} = \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

$$\rightarrow \boxed{x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}}$$

This gives us a formula to solve any quadratic!

Ex 1.6 • $x^2 + x - 8.75 = 0$ $a=b=1, c=-8.75$

$$\rightarrow x = \frac{-1 \pm \sqrt{1 - 4(-8.75)}}{2} = \frac{-1 \pm \sqrt{1 + 35}}{2}$$

$$= \frac{-1 \pm 6}{2} = -\frac{7}{2} \text{ or } \frac{5}{2}$$

• $4x^2 - 20x + 25 = 0$ $a=4, b=-20, c=25$

$$\rightarrow x = \frac{20 \pm \sqrt{400 - 4(4)(25)}}{16} = \frac{20 \pm \sqrt{0}}{16}$$

$$= \frac{20}{16} = \frac{5}{4}$$

How can we tell how many real solutions we have?

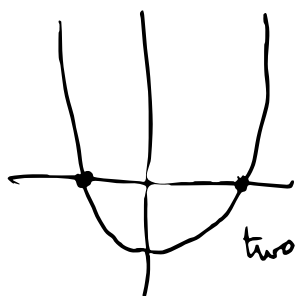
The term that determines this is $\pm \sqrt{b^2 - 4ac}$

• If $b^2 - 4ac > 0$ then get two real roots

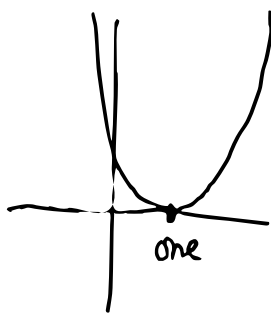
- If $b^2 - 4ac = 0$ then get one real roots.
- If $b^2 - 4ac < 0$ then get no real roots.

$b^2 - 4ac$ is called the discriminant of the quadratic.

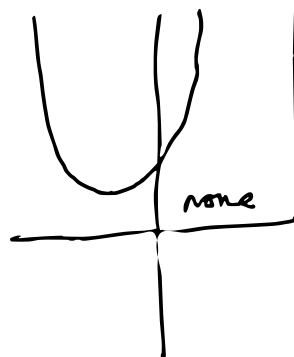
Graphs



$$b^2 - 4ac > 0$$



$$b^2 - 4ac = 0$$



$$b^2 - 4ac < 0$$

Cubic and Quartic polynomials have similar formulae... but messy. Quintics (degree 5) can be proved to not have a general formula.

However, some higher degree polynomials can still be solved by classical techniques!

Ex 1.7 • $x^3 + 2x^2 - 35x = x(x^2 + 2x - 35) = 0$
 $\rightarrow x = 0, -7, 5$

• $x^4 - 7x^2 - 144 = y^2 - 7y - 144 = 0 \quad (y = x^2)$

$\rightarrow y = -9, 16$

$\rightarrow x = \cancel{\pm 9}, \pm 4$

Polynomials 2

Q: Is there a quick way to expand powers of expressions?

A: Yes, for $(a+b)^n$. We'll get there later!

Factorials - • $n! = n(n-1)\dots(3)(2)(1)$ if $n \geq 1$
 (integer)

• $0! = 1$

• $n!$ is the number of ways in which n objects can be arranged:

Boxes $\boxed{2} \quad \boxed{2} \quad \boxed{3} \quad \dots \quad \boxed{n-2} \quad \boxed{n-1} \quad \boxed{n}$
 Objects $\uparrow \quad \uparrow \quad \uparrow \quad \dots \quad \uparrow \quad \uparrow \quad \uparrow$
 $n \times (n-1) \times (n-2) \dots 3 \times 2 \times 1 = n!$

Ex 2.1

$$\begin{aligned} \cdot 4! &= 4 \times 3 \times 2 \times 1 = 24 \\ \cdot \frac{10!}{7!} &= \frac{(10)(9)(8)(7)(6) \dots (\cancel{3})(\cancel{2})(\cancel{1})}{(\cancel{7})(\cancel{6}) \dots (\cancel{3})(\cancel{2})(\cancel{1})} \\ &= (10)(9)(8) = 720 \end{aligned}$$

Permutations

Q: How many ways are there of selecting m objects from n ? (In order)

A: ${}^n P_m = n(n-1)(n-2) \dots (n-(m-1))$

Why? There are n choices for object 1.
 $n-1$ choices for object 2.

$\vdots$
 $n-(m-1)$ choices for object m .

Formula: ${}^n P_m = \frac{n!}{(n-m)!}$ (times top and bottom by $m!$)

Note $\cdot {}^n P_n = \frac{n!}{0!} = n!$

$$\cdot {}^n P_1 = \frac{n!}{(n-1)!} = n$$

Ex 2.2 $\cdot {}^5 P_3 = \frac{5!}{2!} = \frac{120}{2} = 60$

$$\cdot {}^7 P_5 = \frac{7!}{2!} = \frac{5040}{2} = 2520$$

Combinations

What if we don't care about the order of selecting the objects?

In this case, we've overcounted, each permutation of m objects can be reordered in $m!$ ways.

$$\rightarrow {}^nC_m = \frac{1}{m!} {}^nP_m = \frac{n!}{m!(n-m)!}$$

Note $\cdot {}^nC_n = \frac{n!}{n!0!} = 1$

$$\cdot {}^nC_1 = \frac{n!}{(n-1)!} = n$$

Ex 2.3 $\cdot {}^5C_2 = \frac{5!}{2!3!} = \frac{120}{(2)(6)} = 10$

$$\cdot {}^{10}C_8 = \frac{10!}{8!2!} = \frac{(10)(9)}{2} = 45$$

Binomial Expansion

Let's expand $(a+b)^n$ for small n .

Ex 2.4 $(a+b)^1 = a+b$

$$(a+b)^2 = a^2 + 2ab + b^2$$

$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$

$$(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$$

$$\begin{array}{c} 1, 1 \\ 1, 2, 1 \\ 1, 3, 3, 1 \\ 1, 4, 6, 4, 1 \end{array}$$

It turns out that the coefficients are:

$${}^1C_0 = 1 \quad {}^1C_1 = 1$$

$${}^2C_0 = 1 \quad {}^2C_1 = 2 \quad {}^2C_2 = 1$$

$${}^3C_0 = 1 \quad {}^3C_1 = 3 \quad {}^3C_2 = 3 \quad {}^3C_3 = 1$$

$${}^4C_0 = 1 \quad {}^4C_1 = 4 \quad {}^4C_2 = 6 \quad {}^4C_3 = 4 \quad {}^4C_4 = 1$$

In general:

$$\begin{aligned} (a+b)^n &= (a+b)(a+b)\dots(a+b) \\ &= \sum_{m=0}^n {}^nC_m a^m b^{n-m} \end{aligned}$$

Why? To get a term of the form $a^m b^{n-m}$ we must choose the a term from m of the n brackets when multiplying out. There are nC_m ways of doing this!

Ex 35 $\cdot (x+1)^6 = {}^6C_0 x^0 + {}^6C_1 x^1 + {}^6C_2 x^2 + {}^6C_3 x^3 + {}^6C_4 x^4 + {}^6C_5 x^5 + {}^6C_6 x^6$

$$= 1 + 6x + 15x^2 + 20x^3 + 15x^4 + 6x^5 + x^6$$

$$\cdot (2-3x)^4 = {}^4C_0 (2)^0 (-3x)^4 + {}^4C_1 (2)^1 (-3x)^3 + {}^4C_2 (2)^2 (-3x)^2 + {}^4C_3 (2)^3 (-3x)^1 + {}^4C_4 (2)^4 (-3x)^0$$

$$= 81x^4 - 216x^3 + 216x^2 - 96x + 16$$