Level p paramodular congruences of Harder type.

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Outline of talk

1. Harder's conjecture
2. Algebraic modular forms
3. Ibukiyama's correspondence

Let $\Delta \in S_{12}(\mathrm{SL}_2(\mathbb{Z}))$ be the discriminant function. It is a normalized Hecke eigenform with Hecke eigenvalues $\tau(n)$ (the Ramanujan tau function).

Since $\mathrm{ord}_{691}\left(\frac{\zeta(12)}{\pi^{12}}\right) > 0$ we observe a congruence with the weight 12 Eisenstein series:

Ramanujan

For all $n \geq 1$:

$$\tau(n) \equiv \sigma_{11}(n) \bmod 691.$$

Thus for all primes p:

$$\tau(p) \equiv 1 + p^{11} \bmod 691.$$

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G. Harder predicts an analogue of such congruences for Siegel cusp forms.

Harder's conjecture

Let $f \in S_{j+2k-2}(\mathrm{SL}_2(\mathbb{Z}))$ be a normalized Hecke eigenform with Hecke eigenvalues a_n. Suppose there exists a "large prime" λ such that $\mathrm{ord}_\lambda \left(\frac{L(f,j+k)}{\Omega}\right) > 0$ (here Ω is a canonical period).

Then there exists a Hecke eigenform $F \in S_{j,k}(\mathrm{Sp}_4(\mathbb{Z}))$ with Hecke eigenvalues b_n such that for all primes p:

$$b_p \equiv a_p + p^{j+k-1} + p^{k-2} \bmod \Lambda$$

where $\Lambda | \lambda$ in $\mathbb{Q}_F\mathbb{Q}_f$.

A huge part of my PhD to studying the following version of the conjecture for prime level:

Harder's conjecture - level p paramodular version

Let $f \in S_{j+2k-2}^{new}(\Gamma_0(p))$ be a normalized Hecke eigenform with Hecke eigenvalues a_n. Suppose there exists a "large prime" λ such that $\mathrm{ord}_\lambda\left(\frac{L(f,j+k)}{\Omega}\right) > 0$ (here Ω is a canonical period).

Then there exists a Hecke eigenform $F \in S_{j,k}^{new}(K(p))$ with Hecke eigenvalues b_n such that for all primes $q \neq p$:

$$b_q \equiv a_q + q^{j+k-1} + q^{k-2} \bmod \Lambda$$

where $\Lambda | \lambda$ in $\mathbb{Q}_F\mathbb{Q}_f$ and Λ doesn't divide p.

Known numerical evidence

- Level 1: Faber, Van der Geer (see 1-2-3 of modular forms).
- Level 2: Bergstrom et al (method specific to level 2).
- Other levels: Rare.

Outline of talk

Notation:

- $G/\mathbb{Q}$ - connected reductive group such that $G(\mathbb{R})$ is compact modulo center,
- V - finite dimensional representation of G,
- K_f - compact open subgroup of $G(\mathbb{A}_f)$.

Definition

The space of algebraic modular forms $\mathcal{A}(G, K_f, V)$ consists of functions $f : G(\mathbb{A}_f) \to V$ such that $f(\gamma g k) = \gamma \cdot f(g)$ for all $\gamma \in G(\mathbb{Q}), g \in G(\mathbb{A}_f), k \in K_f$.

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Fact 1

The "modular curve" $G(\mathbb{Q})\backslash G(\mathbb{A}_f)/K_f$ is finite. Its size h is called the class number of G with respect to K_f.

Let $z_1, z_2, ..., z_h$ be representatives for the above. Then it is clear that $f \in \mathcal{A}(G, K_f, V)$ is fully determined by the values $f(z_1), f(z_2), ..., f(z_h)$.

Question: Can the values $f(z_i) \in V$ be arbitrary?

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Theorem

The map $f \mapsto (f(z_1), f(z_2), ..., f(z_h))$ gives an isomorphism:

$$\mathcal{A}(G, K_f, V) \cong \oplus_{i=1}^{h} V^{\Gamma_i},$$

where $\Gamma_i = G(\mathbb{Q}) \cap z_i K_f z_i^{-1}$.

Note: This shows finite dimensionality of the spaces $\mathcal{A}(G, K_f, V)$ without having to use analytical tools.

Fact 2

Each Γ_i is a finite group.

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Hecke operators

Choose $\mu \in G(\mathbb{A}_f)$ and consider the decomposition:

$$K_f \mu K_f = \coprod \mu_i K_f$$

Definition

The Hecke operator T_μ acts on $\mathcal{A}(G, K_f, V)$ via:

$$T_\mu(f)(g) = \sum f(g\mu_i).$$

This sum makes sense since:

Fact 3

Any such decomposition of $K_f \mu K_f$ has finitely many μ_i's.

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Outline of talk

Let D be a definite quaternion algebra over $\mathbb{Q}$, ramified at a single prime p.

Consider the quaternionic unitary similitude group:

$$\mathrm{GU}_2(D) = \{g \in \mathrm{M}_2(D) \mid g\bar{g}^T = \mu I, \mu \in \mathbb{Q}^\times\}.$$

Then GSp_4 and GU_2 are inner forms and $\mathrm{GU}_2(D_\infty) = \mathrm{GU}_2(\mathbb{H})$ is compact modulo center.

Ibukiyama's (conjectural) correspondence

There exists an isomorphism of Hecke modules:

$$S^{\mathrm{new}}_{j,k}(K(p)) \cong \mathcal{A}^{\mathrm{new}}(\mathrm{GU}_2(D), U, V_{j,k-3})$$

where U and $V_{j,k-3}$ can be made explicit.

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Hecke operators for $q \neq p$:

Fix $\mathrm{GU}_2(\mathbb{Q}_q) \cong \mathrm{GSp}_4(\mathbb{Q}_q)$ and choose $u \in \mathrm{GU}_2(\mathbb{A}_f)$ such that $u_q \mapsto \mathrm{diag}(1, 1, q, q)$ and $u_r = \mathrm{id}$ for $r \neq q$.

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What is known about this space of algebraic modular forms?

- $h = 1$ if and only if $p = 2, 3, 5, 7, 11$ (Ibukiyama).
- When $h = 1$ I have explicit algorithms to compute:
  - The group $\Gamma = \mathrm{GU}_2(D) \cap U$.
  - The Hecke representatives for T_u when $q \neq p$.

So at levels $2, 3, 5, 7, 11$ I can easily compute Hecke eigenvalues for $F \in S_{j,k}^{\text{new}}(K(p))$ (using a trace formula by Dummigan), hence I can test Harder's congruence.

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Example for level 3

Consider the unique normalized eigenform $f \in S_{16}^{\text{new}}(\Gamma_0(3))$. It has $a_2 = -234$.

Magma tells us that $\text{ord}_{109}\left(\frac{L(f,10)}{\Omega}\right) = 1$.

It turns out that $\dim(S_{2,8}^{\text{new}}(K(3))) = 1$.

Using my algorithms and the trace formula we get $\text{tr}(T_u) = -\frac{39}{8}$ so that $b_2 = -2^6\left(\frac{39}{8}\right) = -312$.

Observe:

$$-312 \equiv -234 + 2^9 + 2^6 \bmod 109.$$

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What next?

- Test Harder's conjecture for cases where $\dim(S_{j,k}^{\text{new}}(K(p))) \geq 2$ or $h \neq 1$. Both of these provide a tougher challenge computationally.
- Explain theoretically why we expect to find such congruences for paramodular forms as opposed to other level p structures.
- Find congruences of "local origin" between level 1 and level p, where instead of looking for large primes dividing an L-value we look for them dividing a single Euler factor.

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Thank you for listening!