The Eulerian quandry

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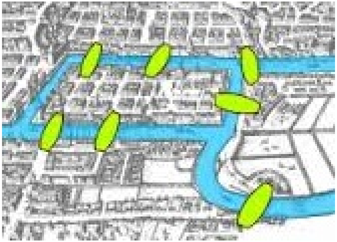
8th April 2014

The Bridges of Konigsberg problem

Konigsberg (now called Kaliningrad) was an old city in Germany. But not just any old city...it boasted a tough problem. A part of the city had seven bridges crossing the river Pregel.



Question

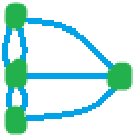
Can I make a journey, crossing every bridge exactly once, and end up back where I started?

The problem was solved in 1735 by famous mathematician Leonhard Euler. His ingenious solution led mathematicians to the invention of a branch of maths known as graph theory.

Essentially Euler noticed that the geometry of the situation (i.e. the sizes of the land, the length of the bridges, the points of connection etc) is irrelevant to the question and could be ignored.

The first ever “graph”

Since we don’t have to care too much about the geometry we can just represent land masses as “dots” and bridges as “lines connecting dots”. Doing so we produce the following picture:



And our question translates into the following:

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Can we start at a dot and travel along each line exactly once, returning to the starting dot?

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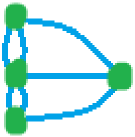

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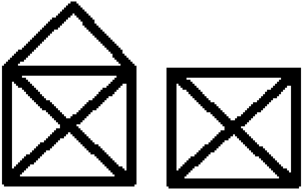
Envelopes

This new question should remind you of another question from your childhood:

Question

Can you draw an open envelope without taking your pen off the paper and without going over the same edge twice? How about a closed envelope?



What is a “graph”

Definition

A graph is:

- A set of “dots”, called vertices
- A set of “lines” connecting the dots, called edges

Graphs may seem simple and pointless but they model many real life situations. For example think about the London underground map, it is a graph!

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Ok so back to our questions. We are searching for paths on graphs where we travel along each edge exactly once. We give a name to graphs where this is possible.

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A graph is called Semi-Eulerian if there is a path that travels along each edge exactly once. A graph is Eulerian if there exists such a path with the same start and end point.

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Euler managed to find a nice result that tells you when a graph is Eulerian or Semi-Eulerian.

Essentially he noticed that if we are not allowed to use an edge twice then each time we travel along an edge to a vertex we must leave by a different edge (except for the start and end vertex).

So we would expect there to be an even number of edges at each vertex (except the start and end vertex).

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Euler's result

The result that Euler proved is the following:

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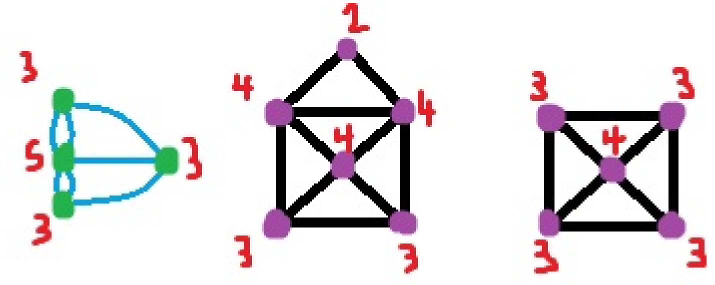

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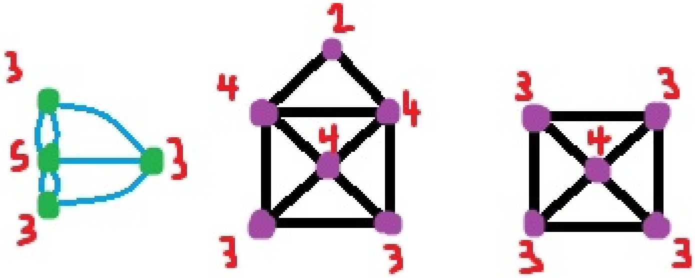

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A few last thoughts...

Here are some puzzles for you to try yourself:

1. Can you “fix” the Konigsberg bridges problem by adding in a bridge? How about if you delete a bridge?
2. Is it possible to use a full set of dominoes to make a loop? (Using the usual rules)
3. Which of these shapes can you draw without taking pen off paper and going over each edge exactly once?