MATH7421 Solutions Week 5

Dirichlet's Unit Theorem

1. Let $1 < w < u$ be a smaller such unit. We can write $w = a + b\sqrt{d}$, with $a, b \in \frac{1}{2}\mathbb{Z}$ (in fact, they are in $\mathbb{Z}$ if $d \equiv 2, 3 \bmod 4$, but we will not need this).

   Since $N(w) = (a + b\sqrt{d})(a - b\sqrt{b}) = \pm 1$ and $w > 1$ we must have that $-1 < a - b\sqrt{b} < 1$. It follows that $0 < T(w) = 2a < 1 + u$, so that $0 < a < \frac{1+u}{2}$. Since there are only finitely many half-integers in any bounded interval, there can only be finitely many possible values of a.

   For each possible value of a we know that b satisfies $N(w) = a^2 - db^2 = \pm 1$, which can have at most 4 solutions. Hence there are only finitely many possibilities for (a, b), hence for w.

   Since there are only finitely many possible units satisfying $1 < w < u$, there must be a smallest such unit...which is either u if no such w exists or is the smallest w otherwise (since any finite subset of $\mathbb{R}$ has a least element!).

2. (a) We have that $v = 2 + \sqrt{3} \in \mathbb{Z}[\sqrt{3}]$ is a unit, since $N(v) = 2^2 - 3 = 1$.

   (b) Suppose that $1 < w < v$ is a smaller such unit and write $w = a + b\sqrt{3}$. Then by the strategy in the notes (or Exercise 1) we have that $0 < a < \frac{1+v}{2} = \frac{3+\sqrt{3}}{2} < 3$. Since $a \in \mathbb{Z}$ we must have that $a = 1$ or $a = 2$.

   If $a = 1$ then $N(w) = a^2 - 3b^2 = 1 - 3b^2 = \pm 1$, which only has solution $b = 0$, so that $w = 1$. This is an invalid solution.

   If $a = 2$ then $N(w) = a^2 - 3b^2 = 4 - 3b^2 = \pm 1$, which only has solution $b = \pm 1$, so that $w = 2 \pm \sqrt{3}$. This is an invalid solution.

   We've shown that there is no unit $1 < w < v$ so that v is the smallest such unit.

   (c) In the notes we have proved that if $K = \mathbb{Q}(\sqrt{d})$ with $d > 1$ square-free and $v > 1$ is the smallest element of $\mathcal{O}_K^\times$ then $\mathcal{O}_K^\times = \{\pm v^m \mid m \in \mathbb{Z}\}$.

   By the work done in (a) and (b) we then have that:

   $$\mathbb{Z}[\sqrt{3}]^\times = \{\pm(2 + \sqrt{3})^m \mid m \in \mathbb{Z}\}.$$

   (d) We saw in the notes that the integer solutions to $x^2 - dy^2 = \pm 1$ are in bijection with units $x + y\sqrt{d} \in \mathbb{Z}[\sqrt{d}]$.

   We can generate three positive integer solutions to $x^2 - 3y^2 = \pm 1$ by using our fundamental unit $v \in \mathbb{Z}[\sqrt{3}]$.

   Since $v = 2 + \sqrt{3}$ we find that $(x, y) = (2, 1)$ is a solution (indeed $2^2 - 3 = 1$).

Since $v^2 = (2+\sqrt{3})^2 = 7+4\sqrt{3}$ we find that $(x,y) = (7,4)$ is a solution (indeed $7^2 - 3\cdot 4^2 = 1$).

Since $v^3 = (2+\sqrt{3})(7+4\sqrt{3}) = 26+15\sqrt{3}$ we find that $(x,y) = (26,15)$ is a solution (indeed $26^2 - 3\cdot 15^2 = 1$).

3. (a) The conjugates of $\sqrt{199}$ are $\pm\sqrt{199}$. These are both real and so we have $(r,s) = (2,0)$ in the notation of Dirichlet's Unit Theorem. The roots of unity in K are ± 1, since $K \subset \mathbb{R}$. Since $r+s-1 = 1$ we have:

$$\mathcal{O}_K^\times = \{\pm v^m \,|\, m \in \mathbb{Z}\}$$

for some fundamental unit $v \in \mathcal{O}_K^\times$.

(b) The conjugates of $\sqrt[3]{-10}$ are $\sqrt[3]{-10}, \zeta\sqrt[3]{-10}, \zeta^2\sqrt[3]{-10}$, where $\zeta = e^{\frac{2\pi i}{3}}$ is a primitive cube root of unity. One of these is real and the other two form a complex conjugate pair, and so we have $(r,s) = (1,1)$ in the notation of Dirichlet's Unit Theorem. The roots of unity in K are ± 1, since $K \subset \mathbb{R}$. Since $r+s-1 = 1$ we have:

$$\mathcal{O}_K^\times = \{\pm v^m \,|\, m \in \mathbb{Z}\}$$

for some fundamental unit $v \in \mathcal{O}_K^\times$.

(c) The conjugates of $\sqrt{2}+\sqrt{3}$ are $\sqrt{2}+\sqrt{3}, \sqrt{2}-\sqrt{3}, -\sqrt{2}+\sqrt{3}, -\sqrt{2}-\sqrt{3}$. These are all real and so we have $(r,s) = (4,0)$ in the notation of Dirichlet's Unit Theorem. The roots of unity in K are ± 1, since $K \subset \mathbb{R}$. Since $r+s-1 = 3$ we have

$$\mathcal{O}_K^\times = \{\pm v_1^{m_1} v_2^{m_2} v_3^{m_3} \,|\, m_1, m_2, m_3 \in \mathbb{Z}\}$$

for some fundamental units $v_1, v_2, v_3 \in \mathcal{O}_K^\times$.

(d) The conjugates of ζ are $\zeta, \zeta^2, \zeta^3, \zeta^4$. These are two complex conjugate pairs and so $(r,s) = (0,2)$ in the notation of Dirichlet's Unit Theorem. The roots of unity in K are $\pm\zeta^i$ for $i = 0,1,2,3,4$. Since $r+s-1 = 1$ we have:

$$\mathcal{O}_K^\times = \{\pm \zeta^i v^m \,|\, i \in \{0,1,2,3,4\}, m \in \mathbb{Z}\}$$

for some fundamental unit $v \in \mathcal{O}_K^\times$.

4. Let $K = \mathbb{Q}(\alpha)$ for some algebraic number α. By Dirichlet's Unit Theorem we have that

$$\mathcal{O}_K^\times = \{\zeta v_1^{m_1} v_2^{m_2} \ldots v_{r+s-1}^{m_{r+s-1}} \,|\, \zeta \text{ is a root of unity in } K \text{ and } m_i \in \mathbb{Z}\}$$

where r is the number of real conjugates of α and $2s$ is the number of complex conjugates.

In order for the unit group to be finite we would require $r+s-1 = 0$, i.e. $r+s = 1$. Since $r, s \geq 0$ this implies that either $(r,s) = (1,0)$ or $(r,s) = (0,1)$.

In the first case we have that the degree of the number field is $[K:\mathbb{Q}] = 1$, which implies that $K = \mathbb{Q}$ (since α must be the root of a linear polynomial, hence be rational).

In the second case we have that the degree of the number field is $[K:\mathbb{Q}] = 2$, which implies that $K = \mathbb{Q}(\sqrt{d})$ for some square-free integer $d \neq 0, 1$ (by exercise 5 in Week 2, or by noting that α is the root of a quadratic polynomial and completing the square). If $d > 0$ then $\sqrt{d}$ has two real conjugates and so we must have that $d < 0$ (in which case $\sqrt{d}$ has two complex conjugates, as required).

5. (a) We know that the solutions to $x^2 - dy^2 = \pm 1$ are in bijection with units in $\mathbb{Z}[\sqrt{d}]$, which we've given are all of the form $\pm(\alpha + \beta\sqrt{d})^n$ for some $n \in \mathbb{Z}$.

Suppose that the unit $(\alpha + \beta\sqrt{d})^n = x + y\sqrt{d}$ corresponds to the solution (x, y). Then a simple calculation shows that the unit

$$(\alpha + \beta\sqrt{d})^{-n} = N(\alpha + \beta\sqrt{d})^{-n}(\alpha - \beta\sqrt{d})^n = (\pm 1)^{-n}(x - y\sqrt{d})$$

corresponds to the solution $(x, -y)$ or $(-x, y)$ (depending on the value of the norm and the parity of n). Thus the four units $\pm(\alpha + \beta\sqrt{d})^n$ and $\pm(\alpha - \beta\sqrt{d})^{-n}$ give rise to the four solutions $(\pm x, \pm y)$.

Now we only need to study the solutions arising from $(\alpha + \beta\sqrt{d})^n$ with $n \geq 0$. The first such unit is $(\alpha + \beta\sqrt{d})^0 = 1$, leading to solution $(a_0, b_0) = (1, 0)$. Suppose that $(\alpha + \beta\sqrt{d})^n = (a_n + b_n\sqrt{d})$. Then

$$(\alpha + \beta\sqrt{d})^{n+1} = (\alpha + \beta\sqrt{d})(a_n + b_n\sqrt{d}) = (\alpha a_n + d\beta b_n) + (\beta a_n + \alpha b_n)\sqrt{d}.$$

This tells us that consecutive solutions are related by the recursion

$$(a_{n+1}, b_{n+1}) = (a_n + 2b_n, a_n + b_n),$$

as required.

(b) The first positive integer solution is $(8, 3)$ and so we run the recursion $(a_{n+1}, b_{n+1}) = (8a_n + 21b_n, 3a_n + 8b_n)$ with starting value $(a_0, b_0) = (1, 0)$ to get the three new solutions $(a_1, b_1) = (8, 3), (a_2, b_2) = (127, 48), (a_3, b_3) = (2024, 765)$ (indeed, one can check that these are all valid solutions to the equation $x^2 - 7y^2 = \pm 1$).

6. (a) If $n = x^2 - dy^2$ for some $x, y \in \mathbb{Z}$ then $n = (x + y\sqrt{d})(x - y\sqrt{d}) = N(x + y\sqrt{d})$ in the ring $\mathbb{Z}[\sqrt{d}]$.

Now let $u \in \mathbb{Z}[\sqrt{d}]^\times$ satisfy $N(u) = 1$ and note that $N(u(x+y\sqrt{d})) = N(u)N(x+y\sqrt{d}) = x^2 - dy^2 = n$. Writing $u(x+y\sqrt{d}) = z + w\sqrt{d}$ we see that $z^2 - dw^2 = n$, so that (z, w) is an integer solution to the equation. Two units $u_1, u_2 \in \mathbb{Z}[\sqrt{d}]^\times$ give different solutions if and only if $u_1(x + y\sqrt{d}) \neq u_2(x + y\sqrt{d})$, i.e. $u_1 \neq u_2$.

We are now done if we can produce infinitely many distinct units of norm 1. To do this, let $v > 1$ be a fundamental unit for $\mathbb{Z}[\sqrt{d}]$. Then $\{v^2, v^4, v^6, ...\}$ is a set of distinct units that are greater than 1 and have norm $N(v^{2m}) = N(v)^{2m} = (\pm 1)^{2m} = 1$.

(b) We have one solution $(x, y) = (3, 1)$, since $3^2 - 2 = 7$. This corresponds to the element $3 + \sqrt{2} \in \mathbb{Z}[\sqrt{2}]$ of norm 7.

To find two more solutions we recall that the fundamental unit of $\mathbb{Z}[\sqrt{2}]$ is $u = 1 + \sqrt{2}$. This has norm $N(u) = -1$ and so we can construct two new solutions by calculating $u^2(3 + \sqrt{2})$ and $u^4(3 + \sqrt{2})$.

Since $u^2(3 + \sqrt{2}) = (1 + \sqrt{2})^2(3 + \sqrt{2}) = (3 + 2\sqrt{2})(3 + \sqrt{2}) = 13 + 9\sqrt{2}$ we get the solution $(x, y) = (13, 9)$ (indeed $13^2 - 2 \cdot 9^2 = 7$).

Since $u^4(3+\sqrt{2}) = (3+2\sqrt{2})^2(3+\sqrt{2}) = (17+12\sqrt{2})(3+\sqrt{2}) = 75+53\sqrt{2}$ we get the solution $(x, y) = (75, 52)$ (indeed $75^2 - 2 \cdot 52^2 = 7$).

7. If $d = m^2$ then $x^2 - dy^2 = x^2 - (my)^2 = (x - my)(x + my) = \pm 1$ implies that $x \pm my \in \{\pm 1\}$. Taking the difference gives $2my \in \{0, \pm 2\}$, so that $my \in \{0, \pm 1\}$.

   If $m \notin \{0, \pm 1\}$ then we must have $y = 0$ and $x^2 = \pm 1$, so that there are exactly two solutions $(x, y) = (\pm 1, 0)$.

   If $m = 0$ then the equation becomes $x^2 = \pm 1$ and this equation has exactly two integer solutions $x = \pm 1$ (or you could argue that there are infinitely many solutions to the original equation, since y can be anything!).

   If $m = \pm 1$ then the equation becomes $x^2 - y^2 = \pm 1$ and we would require $y = 0, \pm 1$. A simple case by case check reveals that there are only four solutions $(x, y) = (\pm 1, 0), (0, \pm 1)$.

8. (a) This is clear since:

$$\begin{aligned}
x + y\left(\frac{1+\sqrt{d}}{2}\right) \in \mathbb{Z}\left[\frac{1+\sqrt{d}}{2}\right]^\times &\iff N\left(x + y\left(\frac{1+\sqrt{d}}{2}\right)\right) = \pm 1 \\
&\iff \left(x + y\left(\frac{1+\sqrt{d}}{2}\right)\right)\left(x + y\left(\frac{1-\sqrt{d}}{2}\right)\right) = \pm 1 \\
&\iff x^2 + xy + \left(\frac{1-d}{4}\right)y^2 = \pm 1.
\end{aligned}$$

   (b) This is very similar to the argument we saw before, for the case of the classical Pell equation. We know that the solutions to $x^2 + xy + \left(\frac{1-d}{4}\right)y^2 = \pm 1$ are in bijection with units in $\mathbb{Z}\left[\frac{1+\sqrt{d}}{2}\right]$, which we've given are all of the form $\pm\left(\alpha + \beta\left(\frac{1+\sqrt{d}}{2}\right)\right)^n$ for some $n \in \mathbb{Z}$.
   Suppose that the unit $\left(\alpha + \beta\left(\frac{1+\sqrt{d}}{2}\right)\right)^n = x + y\left(\frac{1+\sqrt{d}}{2}\right)$ corresponds to the solution (x, y). Then a simple calculation shows that the unit

$$\begin{aligned}
\left(\alpha + \beta\left(\frac{1+\sqrt{d}}{2}\right)\right)^{-n} &= N\left(\alpha + \beta\left(\frac{1+\sqrt{d}}{2}\right)\right)^{-n}\left(\alpha + \beta\left(\frac{1-\sqrt{d}}{2}\right)\right)^{n} \\
&= (\pm 1)^{-n}\left(x + y\left(\frac{1-\sqrt{d}}{2}\right)\right) \\
&= (\pm 1)^{-n}\left((x+y) - y\left(\frac{1+\sqrt{d}}{2}\right)\right)
\end{aligned}$$

   corresponds to the solution $(x + y, -y)$ or $(-(x + y), y)$ (depending on the value of the norm and the parity of n). Thus the four units $\pm\left(\alpha + \beta\left(\frac{1+\sqrt{d}}{2}\right)\right)^n$ and $\pm\left(\alpha + \beta\left(\frac{1-\sqrt{d}}{2}\right)\right)^{-n}$ give rise to the four solutions $\pm(x, y)$ and $\pm((x + y), -y)$.
   Now we only need to study the solutions arising from $\left(\alpha + \beta\left(\frac{1+\sqrt{d}}{2}\right)\right)^n$ with $n \geq 0$. The first such unit is $\left(\alpha + \beta\left(\frac{1+\sqrt{d}}{2}\right)\right)^0 = 1$, leading to solution $(a_0, b_0) = (1, 0)$. Suppose that $\left(\alpha + \beta\left(\frac{1+\sqrt{d}}{2}\right)\right)^n = \left(a_n + b_n\left(\frac{1+\sqrt{d}}{2}\right)\right)$. Then

$$\begin{aligned}
\left(\alpha + \beta\left(\frac{1+\sqrt{d}}{2}\right)\right)^{n+1} &= \left(\alpha + \beta\left(\frac{1+\sqrt{d}}{2}\right)\right)\left(a_n + b_n\left(\frac{1+\sqrt{d}}{2}\right)\right) \\
&= \left(\alpha a_n + \left(\frac{d-1}{4}\right)\beta b_n\right) + (\beta a_n + (\alpha + \beta)b_n)\left(\frac{1+\sqrt{d}}{2}\right).
\end{aligned}$$

This tells us that consecutive solutions are related by the recursion

$$(a_{n+1}, b_{n+1}) \mapsto \left(\alpha a_n + \left(\frac{d-1}{4}\right)\beta b_n, \beta a_n + (\alpha+\beta) b_n\right)$$

as required.

(c) Since the fundamental unit is given and is $\frac{1+\sqrt{5}}{2}$, we know that $\alpha = 0, \beta = 1$. Hence the necessary recursion is $(a_{n+1}, b_{n+1}) = (b_n, a_n + b_n)$. Running the recursion with starting value $(a_0, b_0) = (1, 0)$ gives:

$$\begin{aligned}
(a_1, b_1) &= (0, 1)\\
(a_2, b_2) &= (1, 1)\\
(a_3, b_3) &= (1, 2)\\
(a_4, b_4) &= (2, 3)
\end{aligned}$$

giving three positive integer solutions $(x, y) = (1, 1), (1, 2), (2, 3)$ to $x^2 + xy - y^2 = \pm 1$ (indeed, one can check that these solutions work).

(d) We use induction on n. Note that the base case works since $(a_1, b_1) = (0, 1) = (F_0, F_1)$. Now assume that the result is true for $1 \leq n \leq k$. Then we have that $(a_{k+1}, b_{k+1}) = (b_k, a_k + b_k) = (F_k, F_{k-1} + F_k) = (F_k, F_{k+1})$, so that the result is true for $k+1$. Hence the result is true for all $n \geq 1$.

9. (a) We show that $N(v) = 1$. The conjugates of $\sqrt[2]{3}$ are $\sqrt[2]{3}, \zeta\sqrt[3]{2}, \zeta^2\sqrt[3]{2}^2$, with $\zeta = e^{\frac{2\pi i}{3}} = \frac{-1+\sqrt{-3}}{2}$ a primitive cube root of unity. It follows that:

$$\begin{aligned}
N(1 + \sqrt[3]{2} + \sqrt[3]{2}) &= (1 + \sqrt[3]{2} + \sqrt[3]{2}^2)(1 + \zeta\sqrt[3]{2} + \zeta^2\sqrt[3]{2}^2)(1 + \zeta^2\sqrt[3]{2} + \zeta\sqrt[3]{2}^2)\\
&= (1 + \sqrt[3]{2} + \sqrt[3]{2}^2)((1 + 2\zeta^2 + 2\zeta) + (2 + \zeta^2 + \zeta)\sqrt[3]{2} + (1 + \zeta + \zeta^2)\sqrt[3]{2}^2)\\
&= (1 + \sqrt[3]{2} + \sqrt[3]{2}^2)(-1 + \sqrt[3]{2})\\
&= \sqrt[3]{2}^3 - 1\\
&= 1
\end{aligned}$$

(Here we use the fact that $\zeta^2 + \zeta + 1 = 0$, a fact that we have used many times in previous exercises).

(b) Suppose that there exists a unit satisfying $1 < w < v$. We can write $w = a + b\sqrt[3]{2} + c\sqrt[3]{2}^2$ with $a, b, c \in \mathbb{Z}$. We know that

$$N(w) = (a + b\sqrt[3]{2} + c\sqrt[3]{2}^2)(a + b\zeta\sqrt[3]{2} + c\zeta^2\sqrt[3]{2}^2)(a + b\zeta^2\sqrt[3]{2} + c\zeta\sqrt[3]{2}^2).$$

In the quadratic case we concluded that since $w > 1$, the other term in the norm is between -1 and 1. However, we cannot do this here since the other terms are complex! However, we can say that

$$|N(w)| = |a + b\sqrt[3]{2} + c\sqrt[3]{2}^2| \cdot |a + b\zeta\sqrt[3]{2} + c\zeta^2\sqrt[3]{2}^2| \cdot |a + b\zeta^2\sqrt[3]{2} + c\zeta\sqrt[3]{2}^2| = |\pm 1| = 1.$$

The first term in the product is simply w, and note that:

$$\overline{a + b\zeta\sqrt[3]{2} + c\zeta^2\sqrt[3]{2}^2} = a + b\bar{\zeta}\sqrt[3]{2} + c\bar{\zeta^2}\sqrt[3]{2}^2 = a + b\zeta^2\sqrt[3]{2} + c\zeta\sqrt[3]{2}^2,$$

so that the second and third terms are equal (we have $|z| = |\overline{z}|$ for any $z \in \mathbb{C}$). Let r be this common value.

We now have $wr^2 = 1$ with $w > 1$, so that $r = \frac{1}{\sqrt{w}} < 1$. Hence the second and third terms in the product lie between 0 and 1. Recall that if a complex number satisfies $|z| < R$ for some $R > 0$ then $-R < \mathrm{Re}(z), \mathrm{Im}(z) < R$ (i.e. the coordinates of a point on the circle of radius R lie between $-R$ and R).

Since:

$$\begin{aligned} a + b\zeta\sqrt[3]{2} + c\zeta^2\sqrt[3]{2}^2 &= a + b\left(\frac{-1+\sqrt{-3}}{2}\right)\sqrt[3]{2} + c\left(\frac{-1-\sqrt{-3}}{2}\right)\sqrt[3]{2}^2 \\ &= \left(a - \frac{b}{2}\sqrt[3]{2} - \frac{c}{2}\sqrt[3]{2}^2\right) + \sqrt{3}\left(\frac{b\sqrt[3]{2} - c\sqrt[3]{2}^2}{2}\right) i \end{aligned}$$

we now get the inequalities:

$$\begin{aligned} 1 < a + b\sqrt[3]{2} + c\sqrt[3]{2}^2 &\qquad < 1 + \sqrt[3]{2} + \sqrt[3]{2}^2 \\ -1 < a - \frac{b}{2}\sqrt[3]{2} - \frac{c}{2}\sqrt[3]{2}^2 &\qquad < 1 \\ -1 < \sqrt{3}\left(\frac{b\sqrt[3]{2} - c\sqrt[3]{2}^2}{2}\right) &\qquad < 1 \end{aligned}$$

which rearrange to give:

$$\begin{aligned} 1 < &\qquad a + b\sqrt[3]{2} + c\sqrt[3]{2}^2 &\qquad < 1 + \sqrt[3]{2} + \sqrt[3]{2}^2 \\ -2 < &\qquad 2a - b\sqrt[3]{2} - c\sqrt[3]{2}^2 &\qquad < 2 \\ -\frac{2}{\sqrt{3}} < &\qquad b\sqrt[3]{2} - c\sqrt[3]{2}^2 &\qquad < \frac{2}{\sqrt{3}} \end{aligned}$$

Adding the first two inequalities gives

$$-1 < 3a < 3 + \sqrt[3]{2} + \sqrt[3]{2}^2 \approx 5.847,$$

hence $a = 0$ or $a = 1$. Rearranging and adding/subtracting the second two inequalities gives

$$\begin{aligned} \left(2a - 2 - \frac{2}{\sqrt{3}}\right) < &\qquad 2b\sqrt[3]{2} &\qquad < \left(2 + 2a + \frac{2}{\sqrt{3}}\right) \\ -\left(2 + 2a + \frac{2}{\sqrt{3}}\right) < &\qquad -2c\sqrt[3]{2}^2 &\qquad < \left(2 - 2a + \frac{2}{\sqrt{3}}\right) \end{aligned}$$

which tidy up to give:

$$\begin{aligned} -\frac{1}{\sqrt[3]{2}}\left(1 - a + \frac{1}{\sqrt{3}}\right) < &\qquad b &\qquad < \frac{1}{\sqrt[3]{2}}\left(1 + a + \frac{1}{\sqrt{3}}\right) \\ -\frac{1}{\sqrt[3]{2}^2}\left(1 - a + \frac{1}{\sqrt{3}}\right) < &\qquad c &\qquad < \frac{1}{\sqrt[3]{2}^2}\left(1 + a + \frac{1}{\sqrt{3}}\right) \end{aligned}$$

Case 1: If $a = 0$ then we find that

$$\begin{aligned} -\frac{1}{\sqrt[3]{2}}\left(1 + \frac{1}{\sqrt{3}}\right) < &\qquad b &\qquad < \frac{1}{\sqrt[3]{2}}\left(1 + \frac{1}{\sqrt{3}}\right) \\ -\frac{1}{\sqrt[3]{2}^2}\left(1 + \frac{1}{\sqrt{3}}\right) < &\qquad c &\qquad < \frac{1}{\sqrt[3]{2}^2}\left(1 + \frac{1}{\sqrt{3}}\right) \end{aligned}$$

These inequalities tell us that $b = 0, \pm 1$ and $c = 0$. However, this implies that $w = 0, \pm\sqrt[3]{2}$, neither of which are units.

Case 2: If $a = 1$ then we find that

$$\begin{aligned} -\frac{1}{\sqrt[3]{2}\sqrt{3}} < \quad & b & < \frac{1}{\sqrt[3]{2}}\left(2 + \frac{1}{\sqrt{3}}\right) \\ -\frac{1}{\sqrt[3]{2}^2\sqrt{3}} < \quad & c & < \frac{1}{\sqrt[3]{2}^2}\left(2 + \frac{1}{\sqrt{3}}\right) \end{aligned}$$

These inequalities tell us that $b = 0, \pm 1, \pm 2$ and $c = 0, 1$. However, this implies that

$$\begin{aligned} w = & 1, & 1 \pm \sqrt[3]{2}, && 1 \pm 2\sqrt[3]{2}, \\ & 1 + \sqrt[3]{2}^2, & 1 \pm \sqrt[3]{2} + \sqrt[3]{2}^2, && 1 \pm 2\sqrt[3]{2} + \sqrt[3]{2}^2. \end{aligned}$$

The only possibilities satisfying $1 < w < u$ are

$$w = 1 + \sqrt[3]{2}, \qquad 1 + 2\sqrt[3]{2}, \qquad 1 + \sqrt[3]{2}^2, \qquad 1 - \sqrt[3]{2} + \sqrt[3]{2}^2.$$

It can be checked that none of these have norm ± 1, hence none of these elements are units. This proves the claim that there is no $w \in \mathbb{Z}[\sqrt[3]{2}]^\times$ satisfying $1 < w < u$.

(c) This is identical to the proof for quadratic fields. Suppose that $w \in \mathbb{Z}[\sqrt[3]{2}]^\times \backslash \{\pm 1\}$. Then we may assume that $w > 1$, since exactly one of the units $w, -w, w^{-1}, -w^{-1}$ must be greater than 1. Since $w > 1$ we know that there exists $m \geq 1$ such that $v^m \leq w < v^{m+1}$. Then $1 \leq \frac{w}{v^m} < v$. However, $N\left(\frac{w}{v^m}\right) = \frac{N(w)}{N(v)^m} = \pm 1$, so that $\frac{w}{v_m}$ is a unit. Since it's impossible to have $1 < w < v$, we must have $w = 1$, i.e. $w = v^m$. It follows that every unit is of the form $\pm v^m$ for some $m \in \mathbb{Z}$.