Hilbert modular Eisenstein congruences

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(Joint work with C. Hsu and D. Spencer).

Outline of talk

1. Classical Eisenstein congruences
2. Hilbert Eisenstein series
3. Theorem in progress

Consider the discriminant function (here $z \in \mathcal{H}$ and $q = e^{2\pi i z}$):

$$\Delta(z) = q \prod_{n=1}^{\infty} (1 - q^n)^{24} = \sum_{n=1}^{\infty} \tau(n) q^n.$$

Also let $\sigma_{11}(n)$ be the 11-th power divisor sum of n, i.e. $\sigma_{11}(n) = \sum_{d|n} d^{11}$.

Ramanujan's observation

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We can prove this using the theory of modular forms!

Sketch proof

Consider the Eisenstein series:

$$E_{12}(z) = \frac{\zeta(-11)}{2} + \sum_{n=1}^{\infty} \sigma_{11}(n)q^n \in M_{12}(\mathrm{SL}_2(\mathbb{Z})).$$

Since $\mathrm{ord}_{691}\left(\frac{\zeta(-11)}{2}\right) > 0$ we have that E_{12} is a "mod 691 cuspform".

Lift to characteristic zero and get cusp form with same Fourier coefficients mod 691. It must be Δ.

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Why care?

- It's cool...and completely unexpected!
- It links mysterious numbers $\tau(n)$ with less mysterious numbers $\sigma_{11}(n)$.
- Ramanujan's congruence is equivalent to $\mathrm{Cl}(\mathbb{Q}(\zeta_{691}))^{\chi_{691}^{-11}}$ having an element of order 691 (a stronger statement than 691 being an irregular prime).

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Local origin congruences

Let:

- $k \geq 2$ and $N = uv \geq 1$ with u, v coprime.
- ψ, ϕ, mod N Dirichlet characters with conductors u, v.
- $p \nmid N$ be prime.

Suppose that $\chi = \psi\phi$ satisfies $\chi(-1) = (-1)^k$.

Theorem (Spencer, 2018)

Suppose $\mathrm{ord}_\lambda(L(1-k, \phi\overline{\psi})(\phi(p)p^k - \psi(p))) > 0$ for some prime $\lambda \nmid 6Np$ of $\mathbb{Q}(\psi, \phi)$.
Then there exists a normalized eigenform $f \in S_k(\Gamma_0(Np), \chi)$ (away from Np) and a prime $\Lambda \mid \lambda$ of $\mathbb{Q}(\{a_n(f)\}, \psi, \phi)$ such that:

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Main Ingredient - Consider the Eisenstein series $E_k^{\psi,\phi} \in M_k(\Gamma_0(N), \chi)$ and study the constant term of $E_k^{\psi,\phi}(z) - \psi(p)E_k^{\psi,\phi}(pz) \in M_k(\Gamma_0(Np), \chi)$ mod λ, at all cusps.

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Outline of talk

Notation

Fix the following notation:

- F - a totally real number field, degree d, ring of integers $\mathcal{O}$ and different $\mathfrak{d}$.
- $\sigma_1, ..., \sigma_d$ - the real embeddings of F.
- $\mathfrak{t}_1, ..., t_{h^+}$ - representatives of Cl_F^+.
- $\mathfrak{c}_1, ..., \mathfrak{c}_h$ - representatives of Cl_F.
- $k \geq 2$ and $\mathfrak{m}$ a square-free integral ideal of F.
- χ - a narrow ray class character modulo $\mathfrak{m}$ of signature $\mathbf{r} \in \{0, 1\}^d$.

Let $f : \mathcal{H}^d \to \mathbb{C}$ and $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2^+(F)$.

Define:

$$(f|_k\gamma)(\mathbf{z}) = N(\det(\gamma))^{\frac{k}{2}} N(c\mathbf{z}+d)^k f(\gamma\mathbf{z}),$$

where:

- $\gamma\mathbf{z} = \left(\frac{\sigma_i(a)z_i+\sigma_i(b)}{\sigma_i(c)z_i+\sigma_i(d)}\right)_{i=1,\dots,d}$
- $N(c\mathbf{z}+d) = \prod_{i=1}^{d}(\sigma_i(c)z_i + \sigma_i(d))$.

Let $\Gamma \subset \mathrm{GL}_2^+(F)$ be a "nice" arithmetic subgroup and f be holomorphic (also at infinity when $F = \mathbb{Q}$).
Then f is a Hilbert modular form of weight k, level $\mathfrak{m}$ and character χ if $f|_k\gamma = \chi(d)f$ for all $\gamma \in \Gamma$.

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For each $\lambda \in Cl^+(F)$ we have an analogue of $\Gamma_0(M)$:

$$\Gamma_\lambda(\mathfrak{m}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2^+(F) \,\middle|\, {}_{\substack{a,d\in\mathcal{O}, \\ b\in(\mathfrak{d}\mathfrak{t}_\lambda)^{-1}, c\in\mathfrak{m}\mathfrak{d}\mathfrak{t}_\lambda}}, ad - bc \in \mathcal{O}^\times \right\}.$$

Definition

$M_k(\Gamma_\lambda(\mathfrak{m}), \chi)$ is the space of HMF's of weight k, level $\mathfrak{m}$ and character χ.

In order to get a full Hecke module one needs to consider:

$$M_k(\mathfrak{m}, \chi) = \prod_{\lambda \in Cl^+(F)} M_k(\Gamma_\lambda(\mathfrak{m}), \chi).$$

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Similar to classical case, each $f = (f_\lambda) \in M_k(\mathfrak{m}, \chi)$ has:

- Fourier expansions (one for each λ). However the coefficients depend heavily on the choice of $\mathfrak{t}_\lambda$.
- One can fix this and produce canonical constant terms $c_\lambda(0, f)$ and other coefficients $c(\mathfrak{n}, f)$ for integral $\mathfrak{n}$. We can normalize so that $c(\mathcal{O}, f) = 1$.
- We can lift the level by $\mathfrak{p}$. for each $\lambda \in Cl_F^+$ choose $a_\mu \in F^+$ and $\mu \in Cl_F^+$ such that $\mathfrak{p}\mathfrak{t}_\lambda = a_\mu \mathfrak{t}_\mu$.

  Then $f^{(\mathfrak{p})} \in M_k(\mathfrak{m}\mathfrak{p}, \chi)$ with $f_\lambda^{(\mathfrak{p})}(\mathbf{z}) = \left(N(\mathfrak{t}_\lambda \mathfrak{t}_\mu^{-1})\right)^{\frac{k}{2}} f_\mu(a_\mu \mathbf{z})$.

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Let η, ψ be primitive narrow ray class characters modulo $\mathfrak{a}, \mathfrak{b}$ with signatures $\mathbf{q}, \mathbf{r}$. Also suppose that $\chi = \eta\psi$ and $\mathbf{q} + \mathbf{r} \equiv (k, k, ..., k) \bmod (2\mathbb{Z})^d$.

Theorem (Shimura)

There exists a normalized eigenform $E_k(\eta, \psi) \in M_k(\mathfrak{m}, \chi)$ with:

$$c_\lambda(0, E_k(\eta, \psi)) = \delta_{\mathfrak{a},\mathcal{O}} 2^{-d} \eta^{-1}(\mathfrak{t}_\lambda) L(1-k, \psi\overline{\eta})$$
$$c(\mathfrak{n}, E_k(\eta, \psi)) = \sigma_{k-1}^{\eta,\psi}(\mathfrak{n})$$

for each $\mathfrak{n}$.

From now on we assume that $\mathfrak{a}, \mathfrak{b}$ are coprime and $\mathrm{cond}(\chi) = \mathrm{cond}(\eta)\mathrm{cond}(\psi)$.

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The beast

For $\mathbf{z} \in \mathcal{H}^d$ and $s \in \mathbb{C}$ satisfying $\mathrm{Re}(k+2s) > 2$:

$$E_k(\eta,\psi)_\lambda(\mathbf{z},s) = \frac{C\tau(\psi)}{N(\mathfrak{t}_\lambda)^{\frac{k}{2}} N(\mathfrak{b})} \sum_{\mathfrak{c}\in \mathrm{Cl}_F} N(\mathfrak{c})^k \cdot \sum_{\substack{(a,b)\in S_{\lambda,\mathfrak{c}}/U \\ (a,b)\neq(0,0)}} \frac{\mathrm{sgn}(a)^{\mathbf{q}}\eta(a\mathfrak{c}^{-1})\mathrm{sgn}(-b)^{\mathbf{r}}\psi^{-1}(-b\mathfrak{b}\mathfrak{d}\mathfrak{t}_\lambda\mathfrak{c}^{-1})}{N(a\mathbf{z}+b)^k|a\mathbf{z}+b|^{2s}}$$

Need: constant terms of $E_k(\eta,\psi)$ and $E_k^{(\mathfrak{p})}(\eta,\psi)$, at all cusps.

Theorem (Ozawa)

Each $A \in \mathrm{SL}_2(F)^d$ has $\delta \in \{0,1\}$ and $1 \le i \le h$ such that:

$$c_\lambda(0, E_k(\eta,\psi)|A) = \frac{\delta}{2^d}\frac{\tau(\eta\psi^{-1})}{\tau(\psi^{-1})}\left(\frac{N(\mathfrak{c}_i)}{N(\mathfrak{a})}\right)^k \mathrm{sgn}(-\gamma)^{\mathbf{q}}\eta(\gamma(\mathfrak{b}\mathfrak{d}\mathfrak{t}_\lambda\mathfrak{c}_i)^{-1}) \cdot \mathrm{sgn}(\alpha)^{\mathbf{r}}\psi^{-1}(\alpha\mathfrak{c}_i^{-1})L(1-k,\psi\overline{\eta}).$$

Theorem (F., Hsu, Spencer)

As above but also with $\mathfrak{p}' \in \{\mathcal{O},\mathfrak{p}\}$:

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Outline of talk

"Theorem in progress"

Let $\mathfrak{p} \nmid \mathfrak{m}$ be a prime ideal of $\mathcal{O}$ and suppose that $\mathrm{ord}_\lambda(L(1-k, \psi\overline{\eta})(\psi(\mathfrak{p})N(\mathfrak{p})^k - \eta(\mathfrak{p}))) > 0$ for some prime $\lambda \nmid ?\mathfrak{m}\mathfrak{p}$ of $\mathbb{Q}(\eta, \psi)$.

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Let $E_k^{\eta,\psi} = E_k(\eta,\psi) - \eta(\mathfrak{p})E_k^{(\mathfrak{p})}(\eta,\psi) \in M_k(\mathfrak{m}\mathfrak{p},\chi)$.
Our constant term formula gives the following.

Lemma

Each $A \in \mathrm{SL}_2(F)^d$ has $\delta \in \{0,1\}$ and $1 \le i \le h$ such that:

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$$\cdot\, \mathrm{sgn}(\alpha)^{\mathbf{r}}\psi^{-1}(\alpha\mathfrak{c}_i^{-1})L(1-k,\eta^{-1}\psi)\frac{(\psi(\mathfrak{p})N(\mathfrak{p})^k - \eta(\mathfrak{p}))}{N(\mathfrak{p})^k}.$$

So $E_k^{\eta,\psi}$ is a mod Λ cuspform...but how to lift?

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$$\cdot\,\mathrm{sgn}(\alpha)^{\mathbf{r}}\psi^{-1}(\alpha\mathfrak{c}_i^{-1})L(1-k,\eta^{-1}\psi)\frac{(\psi(\mathfrak{p})N(\mathfrak{p})^k - \eta(\mathfrak{p}))}{N(\mathfrak{p})^k}.$$

So $E_k^{\eta,\psi}$ is a mod Λ cuspform...but how to lift?

"Proof in progress"

Let $E_k^{\eta,\psi} = E_k(\eta,\psi) - \eta(\mathfrak{p})E_k^{(\mathfrak{p})}(\eta,\psi) \in M_k(\mathfrak{m}\mathfrak{p},\chi)$.
Our constant term formula gives the following.

Lemma

Each $A \in \mathrm{SL}_2(F)^d$ has $\delta \in \{0,1\}$ and $1 \leq i \leq h$ such that:

$$c_\lambda(0, E_k^{\eta,\psi}|A) = \frac{\delta}{2^g}\frac{\tau(\eta\psi^{-1})}{\tau(\psi^{-1})}\left(\frac{N(\mathfrak{c}_i)}{N(\mathfrak{a})}\right)^k \mathrm{sgn}(-\gamma)^{\mathbf{q}}\eta(\gamma(\mathfrak{b}\mathfrak{d}\mathfrak{t}_\lambda\mathfrak{c}_i)^{-1})$$
$$\cdot\,\mathrm{sgn}(\alpha)^{\mathbf{r}}\psi^{-1}(\alpha\mathfrak{c}_i^{-1})L(1-k,\eta^{-1}\psi)\frac{(\psi(\mathfrak{p})N(\mathfrak{p})^k - \eta(\mathfrak{p}))}{N(\mathfrak{p})^k}.$$

So $E_k^{\eta,\psi}$ is a mod Λ cuspform...but how to lift?

Thanks for listening!