Interacting particle systems and Jacobi style identities

Dan Fretwell

22 September 2021.

(Joint work with Márton Balázs and Jessica Jay)

Welcome to my PANTHA talk.

Welcome to my PANT talk.

Welcome to my P talk.

Welcome to my (virtual) P talk.

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Hope you find it interesting!

How did I become a closet probabilist?

What did we do?

What does the identity “mean”?

Where do we want to go from here?

 

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I craved more...so I joined forces with Marton Balázs and his PhD student Jessica Jay.

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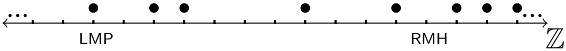
- Probability - Coming right up.

Consider the set of states on $\mathbb{Z}$ that look like:



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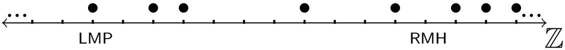

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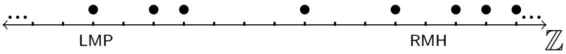

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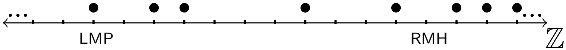

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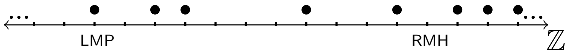

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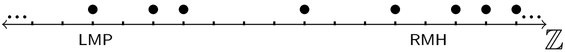

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We have invented the ASEP interacting particle system.

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- Product measure - it decomposes into local distributions $\mu = \otimes \mu_i$ (each μ_i is Bernoulli).

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We would like to know the conditional probability:

$$\nu^{n,c}(\eta) = \mu^c(\eta \,|\, N(\eta) = n),$$

i.e. we need to know $\mu^c(N(\eta) = n)$.

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If $N(\eta)=n$ then:

$$\nu^{n,c}(\eta)=\frac{\mu^c(\eta)}{\mu^c(N(\eta)=n)}.$$

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If η_g has only particles to the right of 0 and only holes to the left:

$$\nu^{0,c}(\eta_g)=\frac{\sum_{m\in\mathbb{Z}} q^{m^2+(1-2c)m}}{\prod_{i\geq 1}(1+q^{2i-1+(1-2c)})(1+q^{2i-1-(1-2c)})}$$

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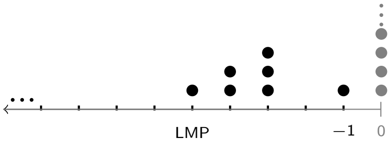

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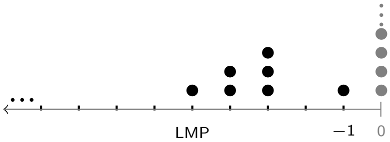

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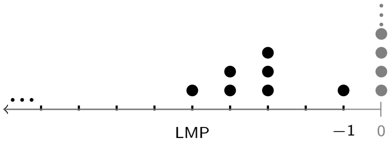

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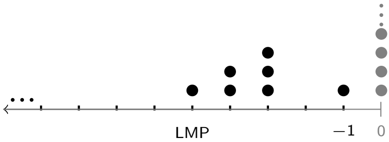

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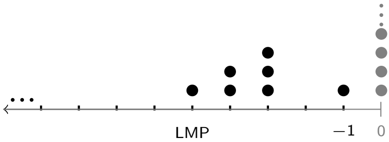

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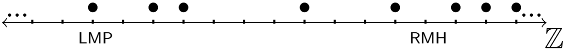
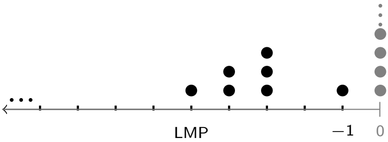
- The empty state ω_g gives $\pi(\omega_g) = \prod_{i \geq 1}(1 - q^{2i})$.

ASEP and AZRP are the same process in disguise!





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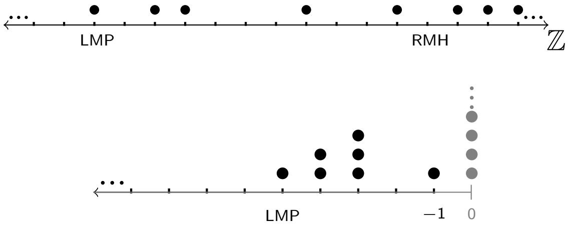

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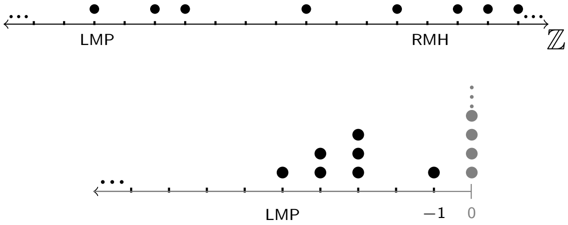

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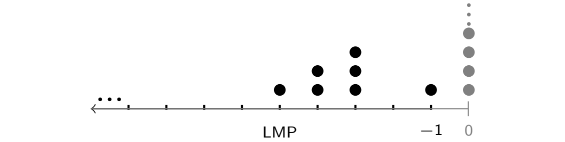

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Can any of these be found using interacting particle systems?

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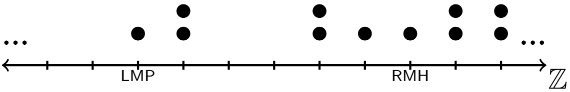

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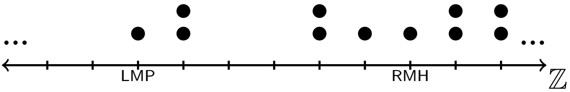

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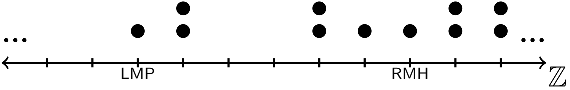

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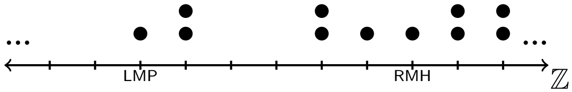

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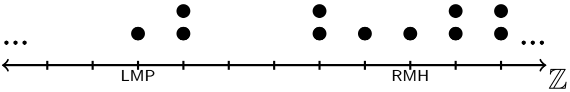

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- Particles are more likely to jump from sites with lots of particles to sites with few particles.
- $p(y,z) = p_{\text{asym}} s(y,z+1) f(y)$ and $q(y,z) = (1-p_{\text{asym}}) s(y+1,z) f(z)$ for some $\frac{1}{2} < p_{\text{asym}} \leq 1$ and functions $f : I \to [0,\infty)$ and $s := I \times I \to [0,\infty)$.

Fact: The general solution to these constraints is a two-parameter family, depending on $\tilde{q} = \frac{q_{\text{asym}}}{p_{\text{asym}}} < 1$ and $t = \sqrt{\frac{p(1,0)q(0,2)}{q(0,1)p(1,1)}} \geq 1$.

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If η_g has 2 particles everywhere to the right of 0 and empty to the left then (after painful manipulation):

$$\nu^{0,c}(\eta_g) = \frac{2\sum_{m\in\mathbb{Z}} \tilde{q}^{m(m+1)} z^{2m}}{\sum_{\epsilon=\pm 1}\prod_{i\geq 1}(1+\epsilon t z\tilde{q}^i + z^2\tilde{q}^{2i})(1+\epsilon t z^{-1}\tilde{q}^i + z^{-2}\tilde{q}^{2i})}$$

Our entire family of processes can be “stood up” as before to make an equivalent AZRP style process on $\mathbb{Z}_{<0}$.

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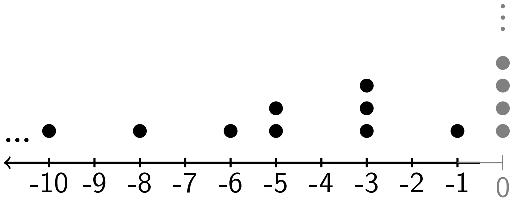

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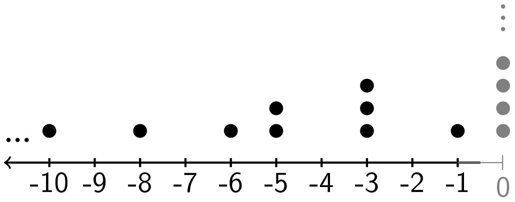

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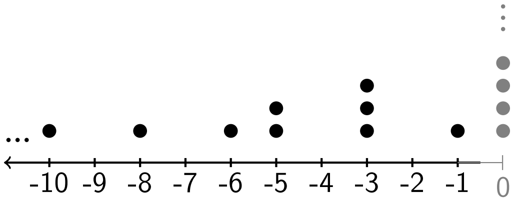

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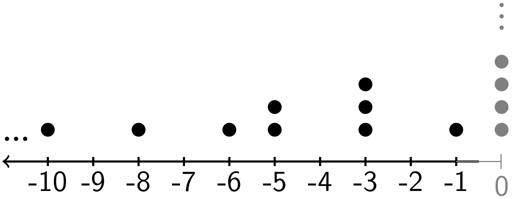

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Much harder to find the stationary distribution...but we did it!

$$\pi(\omega) = \frac{\tilde{q}^{\sum_{i\text{ odd}} i\omega_{-i} + \sum_{i\text{ even}} i(\omega_{-i}-1)}\, t^{2(\sum_{i\text{ odd}} \mathbf{1}(\omega_{-i}\geq 1) - \sum_{i\text{ even}} \mathbf{1}(\omega_{-i}=0))}}{S(\tilde{q}, t)},$$

with $S(\tilde{q}, t)$ the normalising factor (sum the numerator over all states).

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Conjecturally:

$$S(\tilde{q},t) = \frac{1}{\prod_{m\geq 1}(1-\tilde{q}^{2m})} + \frac{\sum_{i\geq 1}\sum_{n\geq i}(-1)^{n-i}\binom{n+i-1}{2i-1}\frac{n}{i}\tilde{q}^{n^2}t^{2i}}{\prod_{m\geq 1}(1-\tilde{q}^{2m})^2}$$

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Partitions of n are in 1-1 correspondence with two row arrays

$$\begin{pmatrix} a_1 & a_2 & \dots & a_s \\ b_1 & b_2 & \dots & b_s \end{pmatrix}$$

such that $n = s + \sum (a_i + b_i)$ and $\{a_i\}, \{b_i\}$ strictly decreasing.

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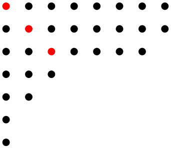

$$\longrightarrow \qquad \begin{pmatrix} 7 & 6 & 4 \\ 6 & 3 & 1 \end{pmatrix}$$

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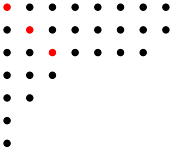

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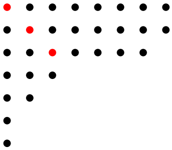

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In general, a Frobenius partition of offset k has row lengths s_1, s_2 satisfying $s_1 - s_2 = k$.

$$\sum_{m\in\mathbb{Z}} \frac{1}{\prod_{i\geq 1}(1-q^i)} q^{\frac{m(m+1)}{2}} z^m = \prod_{i\geq 1}(1+q^i z)(1+q^{i-1}z^{-1}).$$

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As we saw $p_{n,0} = p(n)$. Get other values of k by gluing triangles to Young diagrams.

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This explained our observation about $\prod_{i \geq 1}(1+q^i z)(1+q^{i-1}z^{-1})$.

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By an extension of the "General principle":

$$\prod_{i\geq 1}(1+tz\tilde{q}^i+z^2\tilde{q}^{2i})(1+tz^{-1}\tilde{q}^i+z^{-2}\tilde{q}^{2i}) = \sum_{n,k,d} p_{n,k,d}\tilde{q}^n z^k t^d,$$

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The sum on the RHS counts only those with even offset. In particular $p_{n,0,d}$ is the $\tilde{q}^n t^d$ coefficient of $S(\tilde{q},t)$.

In other words, for each $n, d \geq 0$ we have:

$$p_{n,0,d} = \#\left\{\omega \,\middle|\, \begin{array}{c} \sum_{i \text{ odd}} i\omega_{-i} + \sum_{i \text{ even}} i(\omega_{-i} - 1) = n \\ 2(\sum_{i \text{ odd}} \mathbf{1}(\omega_{-i} \geq 1) - \sum_{i \text{ even}} \mathbf{1}(\omega_{-i} = 0)) = d \end{array}\right\}$$

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Really?! How would anyone have come up with this combinatorially...let alone prove it that way?

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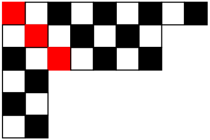
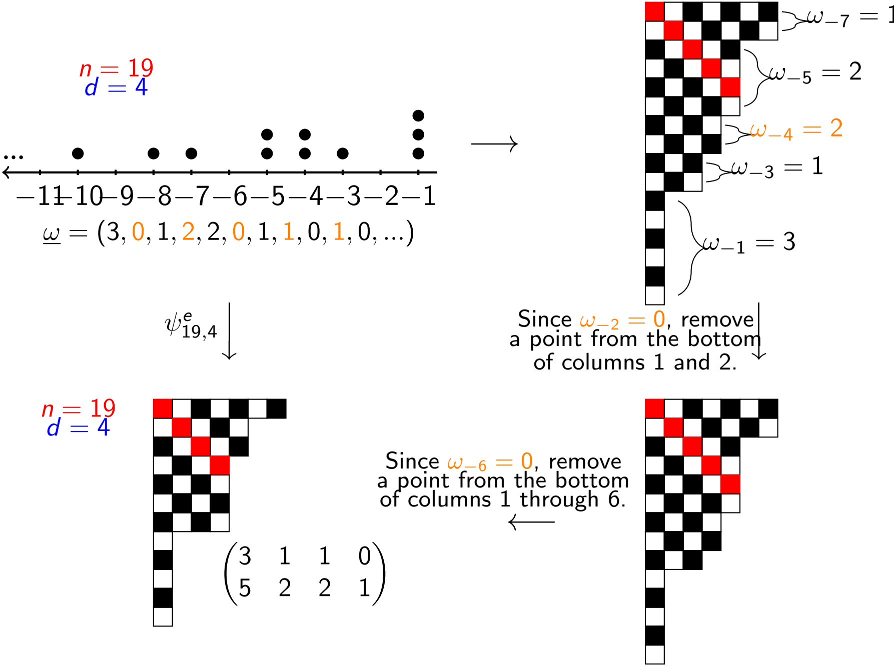


$$\longleftarrow \qquad \begin{pmatrix} 4 & 2 & 2 \\ 2 & 2 & 0 \end{pmatrix}$$



$n = 19$
$d = 4$
-11 -10 -9 -8 -7 -6 -5 -4 -3 -2 -1
$\underline{\omega} = (3, 0, 1, 2, 2, 0, 1, 1, 0, 1, 0, \ldots)$
$\omega_{-7} = 1$
$\omega_{-5} = 2$
$\omega_{-4} = 2$
$\omega_{-3} = 1$
$\omega_{-1} = 3$
$\psi^{e}_{19,4}$
Since $\omega_{-2} = 0$, remove a point from the bottom of columns 1 and 2.
Since $\omega_{-6} = 0$, remove a point from the bottom of columns 1 through 6.
$n = 19$
$d = 4$
$\begin{pmatrix} 3 & 1 & 1 & 0 \\ 5 & 2 & 2 & 1 \end{pmatrix}$

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Thanks for your attention!